A HOLOMORPHIC REPRESENTATION OF THE JACOBI ALGEBRA

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Abstract. A representation of the Jacobi algebra $\mathfrak{h}_1 \rtimes \mathfrak{su}(1,1)$ by first order differential operators with polynomial coefficients on the manifold $\mathbb{C}\times\mathcal{D}_1$ is presented. The Hilbert space of holomorphic functions on which the holomorphic first order differential operators with polynomials coefficients act is constructed.

Contents

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1. Introduction

In this paper we deal with realizations of finite-dimensional Lie algebras by first-order differential operators on homogeneous spaces. Our method, firstly developed in [6], permits to get the holomorphic differential action of the generators of a continuous unitary representation π of a Lie group G with the Lie algebra $\mathfrak{g}$ on a homogeneous space $M = G/H$. We consider homogeneous manifolds realized as Kähler coherent state (CS)-orbits obtained by the action of the representation π on a fixed cyclic vector e_0 belonging to the complex separable Hilbert space $\mathcal{H}$ of the representation [55]. We have applied our method to compact (non-compact) hermitian symmetric spaces in [7] (respectively, [8]) and we have produced simple formulas which show that the differential action of the generators of a hermitian group G on holomorphic functions defined on the hermitian symmetric spaces G/H can be written down as a sum of two terms, one a polynomial P, and the second one a sum of partial derivatives times some polynomials Q-s, the degree of polynomials being less than 3. This is a generalization of the well known realization [47] of the generators $J_{0,+,-}$ of the group $G = \mathrm{SU}(2)$ (and similarly for its non-compact dual $G = \mathrm{SU}(1,1)$) on the homogeneous manifold $G/\mathrm{U}(1)$ by the differential operators $\mathbb{J}_+ = -\frac{\partial}{\partial z}$, $\mathbb{J}_- = -2jz + z^2\frac{\partial}{\partial z}$, $\mathbb{J}_0 = j - z\frac{\partial}{\partial z}$, where the generators verify the commutation relations $[J_0, J_\pm] = \pm J_\pm$, $[J_-, J_+] = -2J_0$ and $J_0 e_0 = -je_0$.

In [10], [13] we have generalized the results of [7], [8] to Kähler CS-orbits of semisimple Lie groups. The differential action of the generators of the groups is of the same type as in the case of hermitian symmetric orbits, i.e. first order differential operators with holomorphic polynomial coefficients, but the maximal degree of the polynomials is grater than 2. We have presented explicit formulas involving the Bernoulli numbers and the structure constants for semisimple Lie groups [10], [13]. The simplest example in which the maximum degree of the polynomials multiplying the derivative is already 3 was worked out in detail in [10], [13], where we have constructed CS on the non-symmetric space $M := \mathrm{SU}(3)/\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1))$.

Let us now recall the standard Segal-Bargmann-Fock [3] realization $a \mapsto \frac{\partial}{\partial z}$; $a^+ \mapsto z$ of the canonical commutation relations (CCR) $[a, a^+] = 1$ on the symmetric Fock space $\mathcal{F}_{\mathcal{H}} := \Gamma^{\mathrm{hol}}(\mathbb{C}, \frac{i}{2\pi}\exp(-|z|^2)dz\wedge d\bar{z})$ attached to the Hilbert space $\mathcal{H} := L^2(\mathbb{R}, dx)$. The Segal-Bargmann-Fock realization can be considered as a representation by differential operators of the real 3-dimensional Heisenberg algebra $\mathfrak{h}_1 \equiv \mathfrak{g}_{HW} =< is1 + z\boldsymbol{a}^+ - \bar{z}\boldsymbol{a} >_{s\in\mathbb{R};z\in\mathbb{C}}$ of the Heisenberg-Weyl group (HW) H_1, where H_n denotes the $(2n+1)$-dimensional HW group. We can look at this construction from group-theoretic point of view, considering the complex number z as local coordinate on the homogeneous manifold $M := H_1/\mathbb{R} \cong \mathbb{C}$. Glauber [30] has attached field coherent states to the points of the manifold M.

In the present paper we are interested in representations of Lie algebras which are semi-direct sum of Heisenberg algebras and semisimple Lie algebras by first order differential operators with holomorphic polynomials coefficients. The most appropriate framework for such an approach is furnished by the so called CS-groups, i.e. groups which admit an orbit which is a complex submanifold of a projective Hilbert space [48],[52]. Indeed, such groups contain all compact groups, all simple hermitian groups, certain solvable groups and also some mixed groups as the semi-direct product of the

HW group and the symplectic group [52]. In reference [11] we have advanced the hypothesis that *the generators of CS-groups admit representations by first order differential operators with holomorphic polynomials coefficients on CS-manifolds.* Here we just present explicit formulas for the simplest example of such a representation of the Lie algebra semi-direct sum of the three-dimensional Heisenberg algebra $\mathfrak{h}_1$ and the algebra of the group $\mathrm{SU}(1,1)$ acting on it in the canonical fashion, $\mathfrak{g}^J_1 := \mathfrak{h}_1 \rtimes \mathfrak{su}(1,1)$, called *Jacobi algebra* (cf. [28] or p. 178 in [52]). The case of the Jacobi algebra $\mathfrak{g}^J_n = \mathfrak{h}_n \rtimes \mathfrak{sp}(n,\mathbb{R})$ is treated separately [14]. Let us remained also that the Jacobi algebra $\mathfrak{g}^J_n$, also denoted $\mathfrak{st}(n,\mathbb{R})$ by Kirillov in §18.4 of [42] or $\mathfrak{tsp}(2n+2,\mathbb{R})$ in [44], is isomorphic with the subalgebra of Weyl algebra A_n (see also [25]) of polynomials of degree maximum 2 in the variables $p_1,\dots,p_n,q_1,\dots,q_n$ with the Poisson bracket, while the Heisenberg algebra $\mathfrak{h}_n$ is the nilpotent ideal isomorphic with polynomials of degree ≤ 1 and the real symplectic algebra $\mathfrak{sp}(n,\mathbb{R})$ is isomorphic to the subspace of symmetric homogeneous polynomials of degree 2. In this paper we study the 6-dimensional Jacobi algebra $\mathfrak{g}^J_1$ and we denote it just $\mathfrak{g}^J$, when there is no possibility of confusion with $\mathfrak{g}^J_n$.

The representations of the Jacobi group were investigated also by the orbit method [42, 43], starting from a matrix representation (see p. 182 in [43]) of the Jacobi algebra $\mathfrak{g}^J$ in [19, 20]. Our method is inspired from the squeezed states of Quantum Optics, see e.g. the reviews [66, 62, 26, 27]. It is well known that for the harmonic oscillator CSs the uncertainties in momentum and position are equal with $1/\sqrt{2}$ (in units of $\hbar$). "The squeezed states" [41, 63, 49, 21, 65, 33, 64] are the states for which the uncertainty in position is less than $1/\sqrt{2}$. The squeezed states are a particular class of "minimum uncertainty states" (MUS) [50], i.e. states which saturates the Heisenberg uncertainty relation. In the present paper we do not insist on the applications of our paper to the squeezed states, the Gaussian states [60, 1], disentangling theorems, i.e. analytic Backer-Campbell-Hausdorff relations defined from a 4×4-matrix representation of the Jacobi algebra, or nonlinear coherent states [62]. Let us just mention that "Gaussian pure states" ("Gaussons") [60] are more general MUSs. In fact, as was shown in [1], these states are CSs based on the manifold $\mathcal{X}^J_n := \mathcal{H}_n \times \mathbb{R}^{2n}$, where $\mathcal{H}_n$ is the Siegel upper half plane $\mathcal{H}_n := \{Z \in M_n(\mathbb{C}) | Z = U + iV, U, V \in M_n(\mathbb{R}), \Im(V) > 0, U^t = U; V^t = V\}$. $M_n(R)$ denotes the $n\times n$ matrices with entries in R, $R = \mathbb{R}$ or $\mathbb{C}$ and X^t denotes the transpose of the matrix X. In [14] we have started the generalization of CSs attached to the Jacobi group $G^J_1 = H_1 \rtimes \mathrm{SU}(1,1)$ to the Jacobi group $G^J_n = H_n \rtimes \mathrm{Sp}(n,\mathbb{R})$. The connection of our construction of coherent states based on $\mathcal{D}^J_n = \mathbb{C}^n \times \mathcal{D}_n$ [14] and the Gaussons of [60] is a subtle one and should be investigated separately. $\mathcal{D}_n$ denotes the Siegel ball $\mathcal{D}_n := \{Z \in M_n(\mathbb{C}) | Z = Z^t,\ 1 - Z\bar{Z} > 0\}$. In §9 we indicate the clue of this connection in the present case, $n = 1$, which is offered by the Kähler-Berndt's construction, shortly sketched in the same section §9. The only physical applications are contained in §8, where we use the expressions of the generators of the Jacobi group G^J_1 to determine the quantum and classical evolution on the manifold $\mathcal{D}^J_1$, generated by a linear Hamiltonian in the generators of the group.

We emphasize that some of the results obtained in this paper, as the reproducing kernel or the group action on the base manifold, can be obtained as particular cases of some of the formulas in Chapter III, Propositions 5.1-5.3 in [59] and §XII.4 in [52].

We also stress that some of the formulas presented here appear in the context of automorphic Jacobi forms [28, 19] -this denomination is inspired by the book [56]. The Jacobi group can be associated (see Chapter 5 entitled *"Kähler's New Poincaré group"* in the article *"Survey of Kähler's mathematical work and some comments"* of R. Berndt and O. Riemenschneider in [40]) with the group G^K investigated by Kähler [37, 38, 39] as a group of the *Universal Theory of Everything*, including relativity, quantum mechanics and even biology. In the paper [37] Kähler has determined the structure of the real 10-dimensional Lie algebra $\mathfrak{g}^K$ of the (*Poincaré* or *New Poincaré*) group G^K and has realized this algebra by differential operators in four **real** variables. However, our approach and the proofs are independent and, we hope, more accessible to people familiar with the coherent state approach in Theoretical Physics and in Mathematical Physics. Moreover, as far as we know, some of the formulas presented in this paper are completely new, e.g. (7.8) expressing the base of polynomials defined on the manifold $\mathcal{D}^J_1$ - the homogeneous space of the Jacobi group G^J_1, acting by biholomorphic maps, or the resolution of unity (7.14)-(7.15).

In order to facilitate the understanding of all subsequent sections, we present in §2 the general setting concerning the CS-groups: §2.1 briefly recalls the definition of CS-groups and §2.2 defines the space of functions, called the symmetric Fock space, on which the the differential operators act (§2.3). However, we shall not enter into a detailed analysis of the root structure of CS-Lie algebras [52], keeping the exposition as elementary as possible. §3 presents the Jacobi algebra $\mathfrak{g}^J$. Perelomov's CS-vectors associated with the Jacobi group G^J_1 (cf. denomination used in [19] or at p. 701 in [52]) are based on the complex homogeneous manifold $M := \mathcal{D}^J_1$. The differential action of the generators of the Jacobi group is given in Lemma 1 of §4. The operators a and a^+ are unbounded operators, but it is enough to work on the dense subspace of smooth vectors of the Hilbert space of the hermitian representation (cf. p. 40 in [52] and also §2.3 of our paper). In Lemma 2 of §5 we calculate the reproducing kernel $K : \mathcal{D}^J_1 \times \bar{\mathcal{D}}^J_1 \rightarrow \mathbb{C}$. Some facts concerning the representations of the HW group H_1 and $\mathrm{SU}(1,1)$ are collected in §6.1. Several relations are obtained in §6.2 as a consequence of the fact that the Heisenberg algebra is an ideal of the Jacobi algebra, and we find how to change the order of the representations of the groups HW and $\mathrm{SU}(1,1)$. Some of the relations presented in §6.2 have appeared earlier in connection with the squeezed states [41] in Quantum Optics [63]. The main result of §6.3 is given in Proposition 1, which expresses the action of the Jacobi group on Perelomov's CS-vectors. Remark 9 establishes the connection of our results in the context of coherent states with those obtained in the theory of automorphic Jacobi forms [28]. In §7.3 we construct the symmetric Fock space attached to the reproducing kernel K from the symmetric Fock spaces associated with the groups HW (cf. §7.1) and $\mathrm{SU}(1,1)$ (cf. §7.2). The G^J_1-invariant Kähler two-form ω, the Liouville form and the equations of geodesics on the manifold $\mathcal{D}^J_1$ are calculated in §7.4. Proposition 2 summarizes all the information obtained in §7 concerning the symmetric Fock space $\mathcal{F}_K$ attached to the reproducing kernel K for the Jacobi group G^J_1, while Proposition 3 gives the continuous unitary holomorphic representation π_K of G^J_1 on $\mathcal{F}_K$. Simple applications to equations of motion on $\mathcal{D}^J_1$ determined by linear Hamiltonians in the generators of the Jacobi group are presented in §8. The equation of

motion is a matrix Riccati equation on the manifold $\mathcal{D}^J_1$. In order to compare our Kähler two-form ω with that given by E. Kähler (see [40], which reproduces [37, 38, 39]) and R. Berndt [17, 19], we express in §9 our ω in coordinates on $\mathcal{D}^J_1$ in appropriate (called in [19] *EZ*) coordinates in $\mathcal{X}^J_1$. The Kähler-Berndt's two-form is in fact the Kähler two-form attached to the manifold $\mathcal{X}^J_1$ on which are based the Gaussons considered in [60] in the case n=1. §10 contains some more remarks referring to the connection between the formulas proved in the present article for the Jacobi algebra and the formalism used in [52] for CS-groups. In order to be self-contained, two formulas referring to the groups HW and SU(1, 1) are proved in the Appendix.

2. Coherent states: the general setting

2.1. **Coherent state groups.** Let us consider the triplet $(G, \pi, \mathcal{H})$, where π is a continuous, unitary representation of the Lie group G on the separable complex Hilbert space $\mathcal{H}$. Let us denote by $\mathcal{H}^\infty$ the *smooth vectors*.

Let us pick up $e_0 \in \mathcal{H}^\infty$ and let the notation: $e_{g,0} := \pi(g).e_0, g \in G$. We have an action $G \times \mathcal{H}^\infty \to \mathcal{H}^\infty$, $g.e_0 := e_{g,0}$. When there is no possibility of confusion, we write just e_g for $e_{g,0}$.

Let us denote by $[\,] : \mathcal{H}^\times := \mathcal{H} \setminus \{0\} \to \mathbb{P}(\mathcal{H}) = \mathcal{H}^\times / \sim$ the projection with respect to the equivalence relation $[\lambda x] \sim [x],\ \lambda \in \mathbb{C}^\times,\ x \in \mathcal{H}^\times$. So, $[.] : \mathcal{H}^\times \to \mathbb{P}(\mathcal{H}),\ [v] = \mathbb{C}v$. The action $G \times \mathcal{H}^\infty \to \mathcal{H}^\infty$ extends to the action $G \times \mathbb{P}(\mathcal{H}^\infty) \to \mathbb{P}(\mathcal{H}^\infty),\ g.[v] := [g.v]$.

For $X \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of the Lie group G, let us define the (unbounded) operator $d\pi(X)$ on $\mathcal{H}$ by $d\pi(X).v := d/dt|_{t=0}\, \pi(\exp tX).v$, whenever the limit on the right hand side exists. We obtain a representation of the Lie algebra $\mathfrak{g}$ on $\mathcal{H}^\infty$, *the derived representation*, and we denote $\boldsymbol{X}.v := d\pi(X).v$ for $X \in \mathfrak{g}, v \in \mathcal{H}^\infty$. Extending $d\pi$ by complex linearity, we get a representation of the universal enveloping algebra of the complex Lie algebra $\mathfrak{g}_\mathbb{C}$ on the complex vector space $\mathcal{H}^\infty$, $d\pi : \mathcal{S} := \mathcal{U}(\mathfrak{g}_\mathbb{C}) \to B_0(\mathcal{H}^\infty)$. Here $B_0(\mathcal{H}^0) \subset \mathcal{L}(\mathcal{H})$, where $\mathcal{H}^0 := \mathcal{H}^\infty$ denotes the subset of linear operators $A : \mathcal{H}^0 \to \mathcal{H}^0$ which have a formal adjoint (cf. p. 29 in [52]).

Let us now denote by H the isotropy group $H := G_{[e_0]} := \{g \in G | g.e_0 \in \mathbb{C}e_0\}$. We shall consider (generalized) coherent states on complex homogeneous manifolds $M \cong G/H$ [55], imposing the restriction that M be *a complex submanifold of* $\mathbb{P}(\mathcal{H}^\infty)$. In such a case the orbit M is called a *CS-manifold* and the groups G which generate such orbits are called *CS-groups* (cf. Definition XV.2.1 at p. 650 and Theorem XV.1.1 at p. 646 in [52]), while their Lie algebras are called *CS-Lie algebras*.

The coherent vector mapping is defined locally, on a coordinate neighborhood $\mathcal{V}_0$, $\varphi : M \to \bar{\mathcal{H}},\ \varphi(z) = e_{\bar{z}}$ (cf. [11]), where $\bar{\mathcal{H}}$ denotes the Hilbert space conjugate to $\mathcal{H}$. The vectors $e_{\bar{z}} \in \bar{\mathcal{H}}$ indexed by the points $z \in M$ are called *Perelomov's coherent state vectors*. The precise definition depends on the root structure of the CS-Lie algebras and we do not go into the details here (see [11]), but only in §10 we just specify the root structure according to [52] in the case of the Jacobi algebra.

We use for the scalar product the convention: $(\lambda x, y) = \bar{\lambda}(x, y),\ x, y \in \mathcal{H}, \lambda \in \mathbb{C}$.

2.2. **The symmetric Fock space.** The space of holomorphic functions (in fact, holomorphic sections of a certain G-homogeneous line bundle over M [52], [11]) $\mathcal{F}_\mathcal{H}$ is defined

as the set of square integrable functions with respect to the scalar product

$$(f,g)_{\mathcal{F}_{\mathcal{H}}} = \int_M \bar{f}(z)g(z)d\nu_M(z,\bar{z}), \tag{2.1}$$

$$d\nu_M(z,\bar{z}) = \frac{\Omega_M(z,\bar{z})}{(e_{\bar{z}},e_{\bar{z}})}. \tag{2.2}$$

Here Ω_M is the normalized G-invariant volume form

$$\Omega_M := (-1)^{\binom{n}{2}}\frac{1}{n!}\underbrace{\omega\wedge\ldots\wedge\omega}_{n \text{ times}}, \tag{2.3}$$

and the G-invariant Kähler two-form ω on the $2n$-dimensional manifold M is given by

$$\omega(z) = i\sum_{\alpha,\beta}\mathcal{G}_{\alpha,\beta}dz_\alpha\wedge d\bar{z}_\beta,\ \mathcal{G}_{\alpha,\beta}(z) = \frac{\partial^2}{\partial z_\alpha\partial\bar{z}_\beta}\log(e_{\bar{z}},e_{\bar{z}}). \tag{2.4}$$

It can be shown that (2.1) is nothing else that *Parseval overcompletness identity* [15]

$$(\psi_1,\psi_2) = \int_{M=G/H}(\psi_1,e_{\bar{z}})(e_{\bar{z}},\psi_2)d\nu_M(z,\bar{z}),\ (\psi_1,\psi_2\in\mathcal{H}). \tag{2.5}$$

It can be seen that *relation* (2.1) *(or* (2.5)*) on homogeneous manifolds fits into Rawnsley's global realization* [58] *of Berezin's coherent states on quantizable Kähler manifolds* [15], modulo Rawnsley's "epsilon" function [58, 24], a constant for homogeneous quantization. If (M,ω) is a Kähler manifold and (L,h,∇) is a (quantum) holomorphic line bundle L on M, where h is the hermitian metric and ∇ is the connection compatible with the metric and the complex structure, then $h(z,\overline{z}) = (e_{\bar{z}},e_{\bar{z}})^{-1}$ and the Kähler potential f is $f = -\log h(z)$ (see e.g. [9]).

Let us now introduce the map

$$\Phi:\mathcal{H}^\star\rightarrow\mathcal{F}_{\mathcal{H}}, \Phi(\psi) := f_\psi, f_\psi(z) = \Phi(\psi)(z) = (\varphi(z),\psi)_{\mathcal{H}} = (e_{\bar{z}},\psi)_{\mathcal{H}},\ z\in\mathcal{V}_0, \tag{2.6}$$

where we have identified the space $\overline{\mathcal{H}}$ complex conjugate to $\mathcal{H}$ with the dual space $\mathcal{H}^\star$ of $\mathcal{H}$.

It can be defined a function $K: M\times\overline{M}\rightarrow\mathbb{C}$, which on $\mathcal{V}_0\times\overline{\mathcal{V}}_0$ reads

$$K(z,\overline{w}) := K_w(z) = (e_{\bar{z}},e_{\bar{w}})_{\mathcal{H}}. \tag{2.7}$$

For CS-groups, *the function* K (2.7) *is a positive definite reproducing kernel; the symmetric Fock space* $\mathcal{F}_{\mathcal{H}}$ *(or* $\mathcal{F}_K$*) is the reproducing kernel Hilbert space of holomorphic functions on* M*,* $\mathcal{H}_K\subset\mathbb{C}^M$*, associated to the kernel* K (2.7), and *the evaluation map* Φ *defined in* (2.6) *extends to an isometric* G*-equivariant embedding* $\mathcal{H}^*\rightarrow\mathcal{F}_{\mathcal{H}}$ [11]

$$(\psi_1,\psi_2)_{\mathcal{H}^\star} = (\Phi(\psi_1),\Phi(\psi_2))_{\mathcal{F}_{\mathcal{H}}} = (f_{\psi_1},f_{\psi_2})_{\mathcal{F}_{\mathcal{H}}} = \int_M\overline{f}_{\psi_1}(z)f_{\psi_2}(z)d\nu_M(z). \tag{2.8}$$

Sometimes the kernel K is considered as a *Bergman section* [54] of a certain bundle over $M\times\bar{M}$, firstly considered by Kobayashi [46], see Chapters V-VIII in [51] and Chapter XII in [52].

2.3. **Representation of CS-Lie algebras by differential operators.** Let us consider again the triplet $(G, \pi, \mathcal{H})$. *The derived representation* $d\pi$ is a *hermitian representation* of the semi-group $\mathcal{S} := \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ on $\mathcal{H}^{\infty}$ (cf. p. 30 in [52]). The unitarity and the continuity of the representation π imply that $id\pi(X)|_{\mathcal{H}^{\infty}}$ is essentially selfadjoint (cf. p. 391 in [52]). Let us denote his image in $B_0(\mathcal{H}^{\infty})$ by $\mathbf{A}_M := d\pi(\mathcal{S})$. If $\Phi : \mathcal{H}^* \rightarrow \mathcal{F}_{\mathcal{H}}$ is the isometry (2.6), we are interested in the study of the image of $\mathbf{A}_M$ via Φ as subset in the algebra of holomorphic, linear differential operators, $\Phi\mathbf{A}_M\Phi^{-1} := \mathbb{A}_M \subset \mathfrak{D}_M$.

The set $\mathfrak{D}_M$ (or simply $\mathfrak{D}$) *of holomorphic, finite order, linear differential operators on* M is a subalgebra of homomorphisms $\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_M, \mathcal{O}_M)$ generated by the set $\mathcal{O}_M$ of germs of holomorphic functions of M and the vector fields. We consider also *the subalgebra* $\mathfrak{A}_M$ of $\mathbb{A}_M$ *of differential operators with holomorphic polynomial coefficients.* Let $U := \mathcal{V}_0 \subset M$, endowed with the local coordinates $(z_1, z_2, \cdots, z_n)$. We set $\partial_i := \frac{\partial}{\partial z_i}$ and $\partial^{\alpha} := \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$, $\alpha := (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}^n$. The sections of $\mathfrak{D}_M$ on U are $A : f \mapsto \sum_{\alpha} a_{\alpha}\partial^{\alpha} f$, $a_{\alpha} \in \Gamma(U, \mathcal{O})$, a_{α}-s being zero except a finite number.

For $k \in \mathbb{N}$, let us denote by $\mathfrak{D}_k$ the subset of differential operators of degree $\leq k$. The filtration of $\mathfrak{D}$ induces a filtration on $\mathfrak{A}$.

Summarizing, we have a correspondence between the following three objects:

$$\mathfrak{g}_{\mathbb{C}} \ni X \mapsto \boldsymbol{X} \in \mathbf{A}_M \mapsto \mathbb{X} \in \mathbb{A}_M \subset \mathfrak{D}_M, \text{ differential operator on } \mathcal{F}_{\mathcal{H}}. \tag{2.9}$$

Moreover, it is easy to see [11] that *if* Φ *is the isometry* (2.6)*, then* $\Phi d\pi(\mathfrak{g}_{\mathbb{C}})\Phi^{-1} \subseteq \mathfrak{D}_1$ *and we have*

$$\mathfrak{g}_{\mathbb{C}} \ni X \mapsto \mathbb{X} \in \mathfrak{D}_1;\ \mathbb{X}_z(f_{\psi}(z)) = \mathbb{X}_z(e_{\bar{z}}, \psi) = (e_{\bar{z}}, \boldsymbol{X}\psi), \tag{2.10}$$

where

$$\mathbb{X}_z(f_{\psi}(z)) = \left(P_X(z) + \sum Q^i_X(z)\frac{\partial}{\partial z_i}\right) f_{\psi}(z). \tag{2.11}$$

In [11] we have advanced the hypothesis that for *CS-groups the holomorphic functions* P *and* Q *in* (2.11) *are polynomials*, i.e. $\mathbb{A} \subset \mathfrak{A}_1 \subset \mathfrak{D}_1$.

In this paper we present explicit formulas for (2.11) in case of the simplest example of a mixed group which is a CS-group, the Jacobi group G^J_1. We start with the Jacobi algebra.

3. The Jacobi algebra

The Heisenberg-Weyl group is the group with the 3-dimensional real Lie algebra isomorphic to the Heisenberg algebra

$$\mathfrak{h}_1 \equiv \mathfrak{g}_{HW} =< is1 + xa^+ - \bar{x}a >_{s\in\mathbb{R}, x\in\mathbb{C}}, \tag{3.1}$$

where a^+ (a) are the boson creation (respectively, annihilation) operators which verify the CCR (3.5a).

Let us also consider the Lie algebra of the group SU(1,1):

$$\mathfrak{su}(1,1) =< 2i\theta K_0 + yK_+ - \bar{y}K_- >_{\theta\in\mathbb{R}, y\in\mathbb{C}}, \tag{3.2}$$

where the generators $K_{0,+,-}$ verify the standard commutation relations (3.5b). We consider the matrix realization

$$K_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; K_+ = i\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, K_- = i\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.3}$$

Now let us define the Jacobi algebra as the the semi-direct sum

$$\mathfrak{g}^J_1 := \mathfrak{h}_1 \rtimes \mathfrak{su}(1,1), \tag{3.4}$$

where $\mathfrak{h}_1$ is an ideal in $\mathfrak{g}^J_1$, i.e. $[\mathfrak{h}_1, \mathfrak{g}^J_1] = \mathfrak{h}_1$, determined by the commutation relations:

$$[a, a^+] = 1, \tag{3.5a}$$
$$[K_0, K_\pm] = \pm K_\pm \; , \; [K_-, K_+] = 2K_0, \tag{3.5b}$$
$$[a, K_+] = a^+ \; , \; \left[K_-, a^+\right] = a, \tag{3.5c}$$
$$\left[K_+, a^+\right] = [K_-, a] = 0, \tag{3.5d}$$
$$\left[K_0, \; a^+\right] = \frac{1}{2}a^+, [K_0, a] = -\frac{1}{2}a. \tag{3.5e}$$

4. The differential action

We shall suppose that we know the derived representation $d\pi$ of the Lie algebra $\mathfrak{g}^J_1$ (3.4) of the Jacobi group G^J_1. We associate to the generators a, a^+ of the HW group and to the generators $K_{0,+,-}$ of the group SU(1,1) the operators a, a^+, respectively $\boldsymbol{K}_{0,+,-}$, where $(a^+)^+ = a$, $\boldsymbol{K}^+_0 = \boldsymbol{K}_0, \boldsymbol{K}^+_\pm = \boldsymbol{K}_\mp$, and we impose to the cyclic vector e_0 to verify simultaneously the conditions

$$ae_0 = 0, \tag{4.1a}$$
$$\boldsymbol{K}_- e_0 = 0, \tag{4.1b}$$
$$\boldsymbol{K}_0 e_0 = ke_0; \; k > 0, 2k = 2, 3, ... \tag{4.1c}$$

We consider in (4.1c) the positive discrete series representations D^+_k of SU(1,1) (cf. §9 in [2]).

Perelomov's coherent state vectors associated to the group G^J_1 with Lie algebra the Jacobi algebra (3.4), based on the manifold M:

$$M := H_1/\mathbb{R} \times \mathrm{SU}(1,1)/\mathrm{U}(1), \tag{4.2a}$$
$$M = \mathcal{D}^J_1 := \mathbb{C} \times \mathcal{D}_1, \tag{4.2b}$$

are defined as

$$e_{z,w} := e^{za^+ + w\boldsymbol{K}_+} e_0, \; z \in \mathbb{C}, \; |w| < 1. \tag{4.3}$$

The general scheme (2.9) associates to elements of the Lie algebra $\mathfrak{g}$ differential operators: $X \in \mathfrak{g} \rightarrow \mathbb{X} \in \mathfrak{D}_1$. The space of functions on which these operators act in the case of the Jacobi group will be made precise in §7.

The following lemma expresses the differential action of the generators of the Jacobi algebra as operators of the type $\mathfrak{A}_1$ in two variables on M.

Lemma 1. *The differential action of the generators* (3.5a)-(3.5e) *of the Jacobi algebra* (3.4) *is given by the formulas:*

$$\boldsymbol{a}=\frac{\partial}{\partial z};\ \boldsymbol{a}^+=z+w\frac{\partial}{\partial z}; \tag{4.4a}$$

$$\mathbb{K}_-=\frac{\partial}{\partial w};\ \mathbb{K}_0=k+\frac{1}{2}z\frac{\partial}{\partial z}+w\frac{\partial}{\partial w}; \tag{4.4b}$$

$$\mathbb{K}_+=\frac{1}{2}z^2+2kw+zw\frac{\partial}{\partial z}+w^2\frac{\partial}{\partial w}, \tag{4.4c}$$

where $z\in\mathbb{C}$, $|w|<1$.

Proof. With the definition (4.3), we have the formal relations:

$$a^+e_{z,w}=\frac{\partial}{\partial z}e_{z,w};\ \boldsymbol{K}_+e_{z,w}=\frac{\partial}{\partial w}e_{z,w}.$$

The proof is based on the general formula

$$\mathrm{Ad}(\exp X)=\exp(\mathrm{ad}_X), \tag{4.5}$$

valid for Lie algebras $\mathfrak{g}$, which here we write down explicitly as

$$Ae^X=e^X(A-[X,A]+\frac{1}{2}[X,[X,A]]+\cdots), \tag{4.6}$$

and we take $X=za^++w\boldsymbol{K}_+$ because of the definition (4.3).

1) Firstly we take in (4.6) $A=a$. Then $[X,A]=-z-wa^+$; $[X,[X,A]]=0$, and

$$ae^X=e^X(a+z+wa^+);$$

$$ae^Xe_0=(z+w\frac{\partial}{\partial z})e^Xe_0.$$

2) Now we take in (4.6) $A=\boldsymbol{K}_0$. Then $[X,A]=-\frac{z}{2}a^+-w\boldsymbol{K}_+$; $[X,[X,A]]=0$, and

$$\boldsymbol{K}_0e_{z,w}=(k+\frac{z}{2}\frac{\partial}{\partial z}+w\frac{\partial}{\partial w})e_{z,w}.$$

3) Finally, we take in (4.6) $A=\boldsymbol{K}_-$. We have $[X,A]=-za-2w\boldsymbol{K}_0$, and

$$\begin{aligned}[X,[X,A]] &= [za^++w\boldsymbol{K}_+,-za-2w\boldsymbol{K}_0]\\ &= -z^2[a^+,a]-2zw[a^+,\boldsymbol{K}_0]-wz[\boldsymbol{K}_+,a]-2w^2[\boldsymbol{K}_+,\boldsymbol{K}_0]\\ &= z^2+2zwa^++2w^2\boldsymbol{K}_+.\end{aligned}$$

Using (4.6), we have

$$\begin{aligned}Ae^Xe_0 &= e^X[\boldsymbol{K}_-+(za+2w\boldsymbol{K}_0)+\frac{1}{2}(z^2+2wza^++2w^2\boldsymbol{K}_+)]e_0\\ &= (2wk+\frac{z^2}{2}+wz\frac{\partial}{\partial z}+w^2\frac{\partial}{\partial w})e^Xe_0.\end{aligned}$$

Now we do some general considerations. For $X\in\mathfrak{g}$, let $\boldsymbol{X}.e_z:=\boldsymbol{X}_ze_z$. Then $\boldsymbol{X}.e_{\bar z}=\boldsymbol{X}_{\bar z}e_{\bar z}$.

But $(e_{\bar z},\boldsymbol{X}.e_{\bar z'})=(\boldsymbol{X}^+.e_{\bar z},e_{\bar z'})$ and finally, with equation (2.10), we have

$$\mathbb{X}_{\bar z'}(e_{\bar z},e_{\bar z'})=\mathbb{X}^+_z(e_{\bar z},e_{\bar z'}) \tag{4.7}$$

With observation (4.7) and the previous calculation, Lemma 1 is proved. ■

Comment 1. *We illustrate* (4.7) *for* $\boldsymbol{X} = \boldsymbol{a}$. *Then it can be checked up that*

$$(\bar{z}' + \bar{w}'\frac{\partial}{\partial \bar{z}'})(e_{\bar{z},\bar{w}}, e_{\bar{z}',\bar{w}'}) = \frac{\partial}{\partial z}(e_{\bar{z},\bar{w}}, e_{\bar{z}',\bar{w}'}) = \frac{\bar{z}' + z\bar{w}'}{1 - w\bar{w}'}(e_{\bar{z},\bar{w}}, e_{\bar{z}',\bar{w}'}),$$

where the kernel has the expression (5.3) *calculated below.*

5. The reproducing kernel

Now we calculate the reproducing kernel K on the base manifold $M = \mathcal{D}^J_1$ as the scalar product of two Perelomov's CS-vectors (4.3), taking into account the conditions (4.1) and the orthonormality of the basis of the Hilbert spaces associated with the factors of the Jacobi group.

Lemma 2. *Let* $K = K(\bar{z}, \bar{w}, z, w)$, *where* $z \in \mathbb{C}$, $w \in \mathbb{C}$, $|w| < 1$,

$$K := (e_0, e^{\bar{z}a + \bar{w}\boldsymbol{K}_-} e^{za^+ + w\boldsymbol{K}_+} e_0). \tag{5.1}$$

Then the reproducing kernel is

$$K = (1 - w\bar{w})^{-2k} \exp \frac{2z\bar{z} + z^2\bar{w} + \bar{z}^2 w}{2(1 - w\bar{w})}. \tag{5.2}$$

More generally, the kernel $K : \mathcal{D}^J_1 \times \bar{\mathcal{D}}^J_1 \rightarrow \mathbb{C}$ *is:*

$$K(z, w; \bar{z}', \bar{w}') := (e_{\bar{z},\bar{w}}, e_{\bar{z}',\bar{w}'}) = (1 - w\bar{w}')^{-2k} \exp \frac{2\bar{z}'z + z^2\bar{w}' + \bar{z}'^2 w}{2(1 - w\bar{w}')}. \tag{5.3}$$

Proof. We introduce the auxiliary operators:

$$\boldsymbol{K}_+ = \frac{1}{2}(a^+)^2 + \boldsymbol{K}'_+, \tag{5.4a}$$

$$\boldsymbol{K}_- = \frac{1}{2}a^2 + \boldsymbol{K}'_-, \tag{5.4b}$$

$$\boldsymbol{K}_0 = \frac{1}{2}(a^+a + \frac{1}{2}) + \boldsymbol{K}'_0, \tag{5.4c}$$

which have the properties

$$\boldsymbol{K}'_- e_0 = 0, \tag{5.5a}$$

$$\boldsymbol{K}'_0 e_0 = k' e_0;\ k = k' + \frac{1}{4}; \tag{5.5b}$$

$$[\boldsymbol{K}'_\sigma, a] = \left[\boldsymbol{K}'_\sigma, a^+\right] = 0,\ \sigma = \pm, 0, \tag{5.6a}$$

$$\left[\boldsymbol{K}'_0, \boldsymbol{K}'_\pm\right] = \pm\boldsymbol{K}'_\pm;\ \left[\boldsymbol{K}'_-, \boldsymbol{K}'_+\right] = 2\boldsymbol{K}'_0. \tag{5.6b}$$

Using the fact that $e_{k,k+m}$ *is an orthonormal system* (see also §7.2 and the Appendix), where

$$e_{k,k+m} := a_{km}(\boldsymbol{K}_+)^m e_{k,k};\ a^2_{km} = \frac{\Gamma(2k)}{m!\Gamma(m + 2k)}, \tag{5.7}$$

the relation (see e.g. equation 1.110 in [31])

$$(1-x)^{-q}=\sum_{m=0}^{\infty}\frac{x^m}{m!}\frac{\Gamma(q+m)}{\Gamma(q)}, \tag{5.8}$$

and the orthonormality of the n-particle states (see also §7.1 and the Appendix):

$$|n>=(n!)^{-\frac{1}{2}}(a^+)^n|0>;\ <n',n>=\delta_{nn'}, \tag{5.9}$$

it is proved the relation

$$(e_0,e^{\bar{w}\boldsymbol{K}'_-}e^{w'\boldsymbol{K}'_+}e_0)=(1-w'\bar{w})^{-2k'}. \tag{5.10}$$

We introduce the notation

$$E=E(z,w):=e^{za^++\frac{w}{2}(a^+)^2}=\sum_{p,q\geq 0}\frac{z^p}{p!}\frac{(\frac{w}{2})^q}{q!}(a^+)^{p+2q}. \tag{5.11}$$

With the change of variable: $n:=p+2q$, i.e. $p=n-2q$, equation (5.11) becomes

$$E=\sum_{n\geq 0}\sum_{q=0}^{[\frac{n}{2}]}\frac{z^{n-2q}}{(n-2q)!q!}\left(\frac{w}{2}\right)^q(a^+)^n.$$

Recalling that the Hermite polynomials can be represented as (cf. equation 10.13.9 in [5])

$$H_n(x)=n!\sum_{m=0}^{[\frac{n}{2}]}\frac{(-1)^m(2x)^{n-2m}}{m!(n-2m)!}, \tag{5.12}$$

the expression (5.11) becomes

$$E(z,w)=\sum_{n\geq 0}\frac{i^{-n}}{n!}\left(\frac{w}{2}\right)^{\frac{n}{2}}H_n\left(\frac{iz}{\sqrt{2w}}\right)(a^+)^n. \tag{5.13}$$

Then

$$K:=K(\bar{z},\bar{w};z',w')=(e_{z,w},e_{z',w'})=(e_0,e^{\bar{z}a+\bar{w}\boldsymbol{K}_-}e^{z'a^++w'\boldsymbol{K}_+}e_0).$$

But due to equations (5.4a), (5.4b), K can be written down as

$$K=(e_0,e^{\bar{z}a+\frac{\bar{w}}{2}a^2}e^{\bar{w}\boldsymbol{K}'_-}e^{w'\boldsymbol{K}'_+}e^{z'a^++\frac{w'}{2}(a^+)^2}e_0).$$

Let the notation

$$F:=F(\bar{z}'\bar{w}';z,w)=(e_0,E^+(\bar{z}',\bar{w}')E(z,w)e_0).$$

Because of the orthonormality relation (5.9), $(e_0,a^{n'}(a^+)^ne_0)=n!\delta_{nn'}$, we get:

$$F=\sum\frac{1}{n!}\left(\frac{\bar{w}'w}{4}\right)^{\frac{n}{2}}H_n(-i\frac{\bar{z}'}{\sqrt{2\bar{w}'}})H_n(i\frac{z}{\sqrt{2w}}).$$

We use the summation relation of the Hermite polynomials (Mehler formula, cf. equation 10.13.22 in [5])

$$\sum_{n=o}^{\infty}\frac{(\frac{s}{2})^n}{n!}H_n(x)H_n(y)=\frac{1}{\sqrt{1-s^2}}\exp\frac{2xys-(x^2+y^2)s^2}{1-s^2},\ |s|<1, \tag{5.14}$$

where

$$x=-i\frac{\bar{z}'}{\sqrt{2\bar{w}'}};\ y=i\frac{z}{\sqrt{2w}};\ s=(\bar{w}'w)^{1/2},$$

and we get

$$F=\frac{1}{(1-\bar{w}'w)^{1/2}}\exp\frac{2\bar{z}'z+\bar{z}'^2w+z^2\bar{w}'}{2(1-\bar{w}'w)}.$$

Recalling (5.10), we have

$$K=(1-\bar{w}'w)^{-2k'}F,$$

and finally:

$$(e_{\bar{z},\bar{w}},e_{\bar{z}',\bar{w}'})=(1-w\bar{w}')^{-2k}\exp\frac{2z\bar{z}'+z^2\bar{w}'+\bar{z}'^2w}{2(1-w\bar{w}')}.$$

■

6. The group action on the base manifold

We start this section recalling in §6.1 some useful relations for representations of the groups H_1 and $\mathrm{SU}(1,1)$. Then we obtain formulas (6.14), (6.15) for the change of order of the action of these groups.

6.1. **Formulas for the Heisenberg-Weyl group H_1 and SU$(1,1)$.** Let us recall some relations for the displacement operator:

$$D(\alpha):=\exp(\alpha a^+-\bar{\alpha}a)=\exp(-\frac{1}{2}|\alpha|^2)\exp(\alpha a^+)\exp(-\bar{\alpha}a), \tag{6.1}$$

$$D(\alpha_2)D(\alpha_1)=e^{i\theta_h(\alpha_2,\alpha_1)}D(\alpha_2+\alpha_1),\ \theta_h(\alpha_2,\alpha_1):=\Im(\alpha_2\bar{\alpha}_1). \tag{6.2}$$

Let us denote by S the D^k_+ representation of the group $\mathrm{SU}(1,1)$ and let us introduce the notation $\underline{S}(z)=S(w)$, where w and z, $w\in\mathbb{C}$, $|w|<1$, $z\in\mathbb{C}$, are related by (6.3c), (6.3d). We have the relations:

$$\underline{S}(z) := \exp(z\boldsymbol{K}_+-\bar{z}\boldsymbol{K}_-),\ z\in\mathbb{C}; \tag{6.3a}$$

$$S(w) = \exp(w\boldsymbol{K}_+)\exp(\eta\boldsymbol{K}_0)\exp(-\bar{w}\boldsymbol{K}_-); \tag{6.3b}$$

$$w = w(z)=\frac{z}{|z|}\tanh\left(|z|\right),w\in\mathbb{C},|w|<1; \tag{6.3c}$$

$$z = z(w)=\frac{w}{|w|}\mathrm{arctanh}\left(|w|\right)=\frac{w}{2|w|}\log\frac{1+|w|}{1-|w|}; \tag{6.3d}$$

$$\eta = \log(1-w\bar{w})=-2\log(\cosh\left(|z|\right)). \tag{6.3e}$$

Let us consider an element $g \in \mathrm{SU}(1,1)$,

$$g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \text{ where } |a|^2 - |b|^2 = 1. \tag{6.4}$$

Remark 1. *The following relations hold:*

$$\underline{S}(z)e_0 = (1-|w|^2)^k e_{0,w}, \tag{6.5}$$

$$e_g := S(g)e_0 = \bar{a}^{-2k} e_{0,w=-i\frac{b}{\bar{a}}} = \left(\frac{a}{\bar{a}}\right)^k \underline{S}(z)e_0, \tag{6.6}$$

$$S(g)e_{0,w} = (\bar{a}+\bar{b}w)^{-2k} e_{0,g\cdot w}, \tag{6.7}$$

where $w \in \mathbb{C}$, $|w| < 1$ *and* $z \in \mathbb{C}$ *in* (6.6) *are related by equations* (6.3c), (6.3d), *and the linear-fractional action of the group* $\mathrm{SU}(1,1)$ *on the unit disk* $\mathcal{D}_1$ *in* (6.7) *is*

$$g \cdot w = \frac{a\,w + b}{\bar{b}\,w + \bar{a}}. \tag{6.8}$$

We recall also the following property, which is a particular case of a more general result proved in [12]:

Remark 2. *If* $\underline{S}(z)$ *is defined by* (6.3a), *then:*

$$\underline{S}(z_2)\underline{S}(z_1) = \underline{S}(z_3)e^{i\theta_s \boldsymbol{K}_0}; \tag{6.9a}$$

$$w_3 = \frac{w_1+w_2}{1+\bar{w}_2 w_1}; \tag{6.9b}$$

$$e^{i\theta_s} = \frac{1+w_2\bar{w}_1}{1+w_1\bar{w}_2}, \tag{6.9c}$$

where w_i *and* z_i, $i = 1,2,3$, *in equation* (6.9b) *are related by the relations* (6.3c), (6.3d).

Comment 2. *Note that when* $z_1, z_2 \in \mathbb{R}$, *then* (6.9a) *expresses just the additivity of the "rapidities",*

$$\underline{S}(z_2)\underline{S}(z_1) = \underline{S}(z_2+z_1),$$

while (6.9b) *becomes just the Lorentz composition of velocities in special relativity:*

$$w_3 = \frac{w_1+w_2}{1+w_2 w_1}.$$

6.2. **Holstein-Primakoff-Bogoliubov-type equations.** We recall the *Holstein-Primakoff-Bogoliubov equations* [34],[23] (see also [53]), a consequence of the equation (4.5) and of the fact that the Heisenberg algebra is an ideal in the Jacobi algebra (3.4), as expressed in (3.5c)-(3.5e):

$$\underline{S}^{-1}(z)\,a\,\underline{S}(z) = \cosh(|z|)\,a + \frac{z}{|z|}\sinh(|z|)\,a^+, \tag{6.10a}$$

$$\underline{S}^{-1}(z)\,a^+\,\underline{S}(z) = \cosh(|z|)\,a^+ + \frac{\bar{z}}{|z|}\sinh(|z|)\,a, \tag{6.10b}$$

and the CCR are still fulfilled in the new creation and annihilation operators.

Let us introduce the notation:

$$\tilde{A} = \left(\begin{array}{c} A \\ \bar{A} \end{array} \right); \ \mathcal{D} = \mathcal{D}(z) = \left(\begin{array}{cc} M & N \\ P & Q \end{array} \right), \tag{6.11}$$

where

$$M = \cosh(|z|); \ N = \frac{z}{|z|}\sinh(|z|); \ P = \bar{N}; \ Q = M. \tag{6.12}$$

Note that

$$\mathcal{D}(z) = e^X, \ \text{where } X := \left(\begin{array}{cc} 0 & z \\ \bar{z} & 0 \end{array} \right). \tag{6.13}$$

Remark 3. *With the notation* (6.11), (6.12), *equations* (6.10) *become:*

$$\underline{S}^{-1}(z)\tilde{a}\underline{S}(z) = \mathcal{D}(z)\tilde{a}.$$

Using formula (4.5), we obtain, as a consequence that the HW group is a normal subgroup of the Jacobi group, the relations (6.14), (6.15) (or (6.16)), which allow to interchange the order of the representations of the groups $\mathrm{SU}(1,1)$ and HW:

Remark 4. *If D and $\underline{S}(z)$ are defined by* (6.1), *respectively* (6.3a), *then*

$$D(\alpha)\underline{S}(z) = \underline{S}(z)D(\beta), \tag{6.14}$$

where

$$\beta = \alpha\cosh(|z|) - \bar{\alpha}\frac{z}{|z|}\sinh(|z|) \quad ; \quad \alpha = \beta\cosh(|z|) + \bar{\beta}\frac{z}{|z|}\sinh(|z|); \tag{6.15a}$$

$$\beta = \frac{\alpha - \bar{\alpha}w}{(1-|w|^2)^{1/2}} \quad ; \quad \alpha = \frac{\beta + \bar{\beta}w}{(1-|w|^2)^{1/2}}. \tag{6.15b}$$

With the convention (6.11), *equation* (6.15a) *can be written down as:*

$$\tilde{\beta} = \mathcal{D}(-z)\tilde{\alpha}; \ \tilde{\alpha} = \mathcal{D}(z)\tilde{\beta}. \tag{6.16}$$

Let us introduce the notation

$$\underline{S}(z,\theta) := \exp(2i\theta\boldsymbol{K}_0 + z\boldsymbol{K}_+ - \bar{z}\boldsymbol{K}_-). \tag{6.17}$$

Using (4.5), more general formulas than Holstein-Primakoff-Bogoliubov equations (6.10) can be proved, namely:

$$\underline{S}(z,\theta)^{-1}(z)\, a\, \underline{S}(z,\theta) \ = \ (\mathrm{cs}\,(x) + i\theta\,\frac{\mathrm{si}\,(x)}{x})\, a + z\frac{\mathrm{si}\,(x)}{x}\, a^+, \tag{6.18a}$$

$$\underline{S}(z,\theta)^{-1}(z)\, a^+\, \underline{S}(z,\theta) \ = \ (\mathrm{cs}\,(x) - i\theta\,\frac{\mathrm{si}\,(x)}{x})\, a^+ + \bar{z}\frac{\mathrm{si}\,(x)}{x}\, a, \tag{6.18b}$$

where

$$\mathrm{cs}\,(x) := \left\{ \begin{array}{ll} \cosh(x), & \text{if } \lambda = x^2 > 0, \\ \cos(x), & \text{if } \lambda = -x^2 < 0, \end{array} \right. ; \lambda := |z|^2 - \theta^2, \tag{6.19}$$

and similarly for $\mathrm{si}\,(x)$.

Let us consider $X \in \mathfrak{su}(1,1)$,

$$X = \left(\begin{array}{cc} i\theta & z \\ \bar{z} & -i\theta \end{array} \right), \ \theta \in \mathbb{R}, \ z \in \mathbb{C}. \tag{6.20}$$

Then $g = e^X \in \mathrm{SU}(1,1)$ is an element of the form (6.4), where

$$a = \mathrm{cs}\,(x) + i\theta\,\frac{\mathrm{si}\,(x)}{x},\ \ b = z\frac{\mathrm{si}\,(x)}{x}. \tag{6.21}$$

If $g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathrm{SU}(1,1)$, then equations (6.18) can be written down as

$$S^{-1}(g)\, a\, S(g) \ = \ \alpha\, a + \beta\, a^+, \tag{6.22a}$$
$$S^{-1}(g)\, a^+\, S(g) \ = \ \bar{\beta}\, a + \bar{\alpha}\, a^+, \tag{6.22b}$$

and we have the following (generalized Holstein-Primakoff-Bogoliubov) equations:

Remark 5. *If S denotes the representation of* $\mathrm{SU}(1,1)$*, with the convention* (6.11)*, we have*

$$S^{-1}(g)\, \tilde{a}\, S(g) = g \cdot \tilde{a}. \tag{6.23}$$

Applying again formula (4.5), we obtain a more general formula than (6.14), namely:

$$\underline{S}(z,\theta) D(\alpha) \underline{S}(z,\theta)^{-1} = D(\alpha_1), \tag{6.24}$$

where

$$\alpha_1 = \alpha_1(z,\alpha,\theta) = \alpha\,\mathrm{cs}\,(x) + (i\theta\alpha + z\bar{\alpha})\,\frac{\mathrm{si}\,(x)}{x}. \tag{6.25}$$

Written down in the form similar to (6.14), equation (6.24) reads

$$D(\alpha)\underline{S}(z,\theta) = \underline{S}(z,\theta) D(\beta_1), \tag{6.26}$$

where $\beta_1 = \beta_1(z,\alpha,\theta) = \alpha_1(z,-\alpha,-\theta)$, i.e.

$$\beta_1 = \alpha\,\mathrm{cs}\,(x) - (i\theta\alpha + z\bar{\alpha})\,\frac{\mathrm{si}\,(x)}{x}\ , \alpha = \beta_1\,\mathrm{cs}\,(x) + (i\theta\beta_1 + z\bar{\beta}_1)\,\frac{\mathrm{si}\,(x)}{x}. \tag{6.27}$$

Note that if $\theta = 0$, then $\underline{S}(z,\theta) = \underline{S}(z)$ and β_1 in (6.27) becomes $\beta_1 = \beta$ with β given by (6.15).

We also underline that if $z = 0$ in (6.24), then (6.25) becomes just

$$\alpha_1 = \alpha(\cos(|\theta|) + i\theta\frac{\sin(|\theta|)}{|\theta|}).$$

Summarizing, we rewrite now equation (6.24) in the following useful form:

Remark 6. *In the matrix realization* (3.3)*, equation* (6.24) *can be written down as*

$$S(g) D(\alpha) S^{-1}(g) = D(\alpha_g), \tag{6.28}$$

where (6.25) *has the expression of the natural action of* $\mathrm{SU}(1,1)\times\mathbb{C}\rightarrow\mathbb{C}$*:* $g\cdot\tilde{\alpha} := \alpha_g$*,*

$$\alpha_g = a\,\alpha + b\,\bar{\alpha}, \tag{6.29}$$

and a, b have the expression (6.21).

Let us remark that the commutation relations (3.5c)-(3.5e) between the generators of the groups SU$(1,1)$ and HW were chosen in such a way that the action of the group SU$(1,1)$ on the complex plane $M \approx \mathbb{C} = H_1/\mathbb{R}$ be the natural one, cf. Remark 6. Such a choice of the action of the group SU$(1,1)$ on the group H_1, a normal subgroup of the Jacobi group G^J_1, was inspired from the squeezed states in Quantum Optics (cf. e.g. [53]). If we had started from the natural action of SU$(1,1)$ on $\mathbb{C}$ given in Remark 6, then the commutation relations (3.5c)-(3.5e) would had followed taking the derivatives in (6.23) realized as (6.18) using the development (6.21).

Now we consider the product of two representations D and S and apply Remark 4:

$$D(\alpha_2)\underline{S}(z_2)D(\alpha_1)\underline{S}(z_1) = D(\alpha_2)D(\alpha)\underline{S}(z_2)\underline{S}(z_1), \tag{6.30}$$

where

$$\alpha = \alpha_1 \cosh(|z_2|) + \bar{\alpha}_1 \frac{z_2}{|z_2|}\sinh(|z_2|), \tag{6.31}$$

or

$$\tilde{\alpha} = \mathcal{D}(z)\tilde{\alpha}_1.$$

Equations (6.30) and (6.31) allow to determine

Remark 7. *The action:* $(\alpha_2, z_2) \times (\alpha_1, w_1) = (A, w)$*, where* $z_2, \alpha_{1,2}, A \in \mathbb{C}$*,* $w, w_1 \in \mathcal{D}_1$ *and the variables of type* w *and* z *are related by equations* (6.3c), (6.3d)*, can be expressed as:*

$$A = \alpha_2 + \alpha_1 \cosh|z_2| + \bar{\alpha}_1\frac{z_2}{|z_2|}\sinh|z_2| = \alpha_2 + \frac{\alpha_1 + \bar{\alpha}_1 w_2}{(1-|w_2|^2)^{1/2}}, \tag{6.32a}$$

$$w = \frac{\cosh|z_2|w_1 + \frac{z_2}{|z_2|}\sinh|z_2|}{\frac{\bar{z}_2}{|z_2|}\sinh|z_2|w_1 + \cosh|z_2|} = \frac{w_1 + w_2}{1 + w_1\bar{w}_2}. \tag{6.32b}$$

Equations (6.32) *express the action* $(\alpha_2, w_2) \times (\alpha_1, w_1) = (\alpha_2 + w_2 \circ \alpha_1, w_2 \circ w_1)$*,* $\alpha_{1,2} \in \mathbb{C}, w_{1,2} \in \mathcal{D}_1$*.* (6.32) *can be written down as:*

$$\tilde{A} = \tilde{\alpha}_2 + \mathcal{D}\tilde{\alpha}_1, \tag{6.33a}$$

$$w = \frac{Mw_1 + N}{Pw_1 + Q}. \tag{6.33b}$$

Let us introduce the normalized vectors:

$$\Psi_{\alpha,w} := D(\alpha)S(w)e_0;\ \alpha \in \mathbb{C},\ w \in \mathbb{C}, |w| < 1. \tag{6.34}$$

As a consequence of (6.30), we have:

Remark 8. *The product of the representations* D *and* $\underline{S}$ *acts on the CS-vector* (6.34) *with the effect:*

$$D(\alpha_2)\underline{S}(z_2)\Psi_{\alpha_1,w_1} = J\Psi_{A,w},\ \textit{where}\ J = e^{i(\theta_h(\alpha_2,\alpha)+k\theta_s)}. \tag{6.35}$$

Above (A, w) *are given by* Remark 7*,* $\theta_h(\alpha_2, \alpha)$ *is given by* (6.2) *with* α *given by* (6.31)*, while* θ_s *is given by* (6.9c) *and the dependence* $w_2 = w_2(z_2)$ *is given by equation* (6.3c)*.*

Note also the following important property (6.36), well known in the Quantum Optics of squeezed states (see e.g. equation (20) p. 3219 in [63]):

Comment 3. *The action of the HW group on the ("squeezed") state vector*

$$\underline{\Psi}_{z,\alpha} = \underline{S}(z)D(\alpha)e_0$$

modifies only the part of the HW group. More precisely, we have

$$D(\beta)\underline{\Psi}z,\alpha = e^{i\eta}\underline{\Psi}z,\alpha+\gamma, \ \textit{where } \eta = \Im(\gamma\bar{\alpha}), \tag{6.36}$$

and

$$\gamma = \beta\cosh(|z|) - \bar{\beta}\frac{z}{|z|}\sinh(|z|), \ \textit{or } \tilde{\gamma} = \mathcal{D}(-z)\tilde{\beta}. \tag{6.37}$$

Indeed, we apply formula (6.14):

$$D(\beta)\underline{S}(z)D(\alpha) = \underline{S}(z)D(\gamma)D(\alpha),$$

where γ has the expression (6.37).
Then (6.36) follows. ■

6.3. **The action of the Jacobi group.** Now we find a relation between the (normalized) vector (6.34) and the (unnormalized) Perelomov's CS-vector (4.3), which will be important in the proof of Proposition 1, our main result of this section.

Lemma 3. *The vectors* (6.34), (4.3), *i.e.*

$$\Psi_{\alpha,w} := D(\alpha)S(w)e_0; \ e_{z,w'} := \exp(za^+ + w'\boldsymbol{K}_+)e_0.$$

are related by the relation

$$\Psi_{\alpha,w} = (1-w\bar{w})^k\exp(-\frac{\bar{\alpha}}{2}z)e_{z,w}, \tag{6.38}$$

where $z = \alpha - w\bar{\alpha}$.

Proof. Due to (6.3a), (6.3b), (4.1b) and (4.1c), we have the relations

$$\begin{aligned} S(w)e_0 &= \exp(w\boldsymbol{K}_+)\exp(\eta\boldsymbol{K}_0)\exp(-\bar{w}\boldsymbol{K}_-)e_0 \\ &= \exp(w\boldsymbol{K}_+)\exp(k\ln(1-w\bar{w}))\mathrm{e}_0 \\ &= (1-w\bar{w})^k\exp(w\boldsymbol{K}_+)e_0, \end{aligned}$$

which is also a proof of (6.5).
We obtain successively

$$\begin{aligned} \Psi_{\alpha,w} &= (1-w\bar{w})^kD(\alpha)\exp(w\boldsymbol{K}_+)e_0 \\ &= (1-w\bar{w})^k\exp(-\frac{1}{2}|\alpha|^2)\exp(\alpha a^+)\exp(-\bar{\alpha}a)\exp(w\boldsymbol{K}_+)e_0 \\ &= (1-w\bar{w})^k\exp(-\frac{1}{2}|\alpha|^2)\exp(\alpha a^+)\exp(-\bar{\alpha}a)\exp(w\boldsymbol{K}_+)\exp(\bar{\alpha}a)\exp(-\bar{\alpha}a)e_0 \\ &= (1-w\bar{w})^k\exp(-\frac{1}{2}|\alpha|^2)\exp(\alpha a^+)Ee_0, \end{aligned}$$

where here

$$E := \exp(-\bar{\alpha}a)\exp(w\boldsymbol{K}_+)\exp(\bar{\alpha}a). \tag{6.39}$$

As a consequence of (4.5)

$$\exp(Z)\exp(X)\exp(-Z) = \exp(X + [Z,X] + \frac{1}{2}[Z,[Z,X]] + \cdots),$$

where, if we take $Z = -\bar{\alpha}a;\ X = w\boldsymbol{K}_+$, then

$$[Z,X] = -\bar{\alpha}wa^+; [Z,[Z,X]] = \bar{\alpha}^2 w.$$

We find for E defined by (6.39) the value

$$E = \exp w(\boldsymbol{K}_+ - \bar{\alpha}a^+ + \frac{\bar{\alpha}^2}{2}),$$

and finally

$$\Psi_{\alpha,w} = \exp(-\frac{1}{2}|\alpha|^2)\exp(w\frac{\bar{\alpha}^2}{2})(1-w\bar{w})^k e_{\alpha-w\bar{\alpha},w},$$

i.e. (6.38). ■

Comment 4. *Starting from* (6.38)*, we reobtain the expression* (5.2) *of the reproducing kernel K.*

Indeed, the normalization $(\Psi_{\alpha,w},\Psi_{\alpha,w}) = 1$ implies that

$$(e_{\alpha-w\bar{\alpha},w}, e_{\alpha-w\bar{\alpha},w}) = \exp(|\alpha|^2 - \frac{w}{2}\bar{\alpha}^2 - c.c.)(1-w\bar{w})^{-2k}. \tag{6.40}$$

With the notation: $\alpha - w\bar{\alpha} = z$, we have

$$\alpha = \frac{z + \bar{z}w}{1-\bar{w}w},$$

and then (6.40) can be rewritten as

$$(e_{z,w}, e_{z,w}) = (1-\bar{w}w)^{-2k}\exp\frac{2z\bar{z} + w\bar{z}^2 + \bar{w}z^2}{2(1-w\bar{w})},$$

i.e. we get another proof of (5.2). ■

From the following proposition we can see the holomorphic action of the group Jacobi

$$G_1^J := H_1 \rtimes \mathrm{SU}(1,1), \tag{6.41}$$

on the manifold $\mathcal{D}_1^J$ (4.2b):

Proposition 1. *Let us consider the action $S(g)D(\alpha)e_{z,w}$, where $g \in \mathrm{SU}(1,1)$ has the form* (6.4)*, $D(\alpha)$ is given by* (6.1)*, and the coherent state vector is defined in* (4.3)*. Then we have the formula* (6.42) *and the relations* (6.43), (6.44)-(6.46)*:*

$$S(g)D(\alpha)e_{z,w} = \lambda e_{z_1,w_1},\ \lambda = \lambda(g,\alpha;z,w), \tag{6.42}$$

$$z_1 = \frac{\alpha - \bar{\alpha}w + z}{\bar{b}w + \bar{a}};\ w_1 = g\cdot w = \frac{aw+b}{\bar{b}w+\bar{a}}, \tag{6.43}$$

$$\lambda = (\bar{a} + \bar{b}w)^{-2k}\exp(\frac{z}{2}\bar{\alpha}_0 - \frac{z_1}{2}\bar{\alpha}_2)\exp i\theta_h(\alpha,\alpha_0), \tag{6.44}$$

$$\alpha_0 = \frac{z + \bar{z}w}{1 - w\bar{w}}, \tag{6.45}$$

$$\alpha_2 = (\alpha + \alpha_0)a + (\bar{\alpha} + \bar{\alpha}_0)b. \tag{6.46}$$

Corollary 1. *The action of the 6-dimensional Jacobi group* (6.41) *on the 4-dimensional manifold* (4.2b)*, where* $\mathcal{D}_1 = \mathrm{SU}(1,1)/\mathrm{U}(1)$*, is given by equations* (6.42), (6.43)*. The composition law in* G_1^J *is*

$$(g_1, \alpha_1, t_1) \circ (g_2, \alpha_2, t_2) = (g_1 \circ g_2, g_2^{-1} \cdot \tilde{\alpha}_1 + \alpha_2, t_1 + t_2 + \Im(g_2^{-1} \cdot \alpha_1 \bar{\alpha}_2)), \tag{6.47}$$

where $g \cdot \tilde{\alpha} := \alpha_g$ *is given by* (6.29)*, and if* g *has the form given by* (6.4)*, then* $g^{-1} \cdot \tilde{\alpha} = \alpha_{g^{-1}} = \bar{a}\alpha - b\bar{\alpha}$*.*

Proof of Proposition 1. With Lemma 3, we have $e_{z,w} = \lambda_1 \Psi_{\alpha_0,w}$, where α_0 is given by (6.45) and $\lambda_1 = \exp(\frac{z}{2}\bar{\alpha}_0)(1 - |w|^2)^{-k}$. Then $I := S(g)D(\alpha)e_{z,w}$ becomes successively

$$\begin{aligned} I &= \lambda_1 S(g) D(\alpha) \Psi_{\alpha_0,w} \\ &= \lambda_1 S(g) D(\alpha) D(\alpha_0) S(w) e_0 \\ &= \lambda_2 S(g) D(\alpha_1) S(w) e_0, \end{aligned}$$

where $\alpha_1 = \alpha + \alpha_0$ and $\lambda_2 = \lambda_1 e^{i\theta_h(\alpha_1,\alpha_0)}$. With equations (6.28), (6.29), we have $I = \lambda_2 D(\alpha_2) S(g) S(w) e_0$, where $\alpha_2 = a\alpha_1 + b\bar{\alpha}_1$. But (6.5) implies $I = \lambda_3 D(\alpha_2) S(g) e_{0,w}$, with $\lambda_3 = \lambda_2(1-|w|^2)^k$. Now we use (6.7) and we find $I = \lambda_4 D(\alpha_2) e_{0,w_1}$, where in accord with (6.8) w_1 is given by (6.43) and $\lambda_4 = (\bar{a} + \bar{b}w)^{-2k}\lambda_3$. We rewrite the last equation as $I = \lambda_5 D(\alpha_2) S(w_1) e_0$, where $\lambda_5 = (1 - |w_1|^2)^{-k}\lambda_4$. Then we apply again Lemma 3 and we find $I = \lambda_6 e_{z_1,w_1}$, where $\lambda_6 = \lambda_5(1 - |w_1|^2)^k \exp(-\frac{\bar{\alpha}_2}{2} z_1)$, and $z_1 = \alpha_2 - w_1\bar{\alpha}_2$. Proposition 1 is proved. ∎

Remark 9. *Combining the expressions* (6.43)-(6.46)*, the factor* λ *in* (6.42) *can be written down as*

$$\lambda = (\bar{a} + \bar{b}w)^{-2k} \exp(-\lambda_1), \tag{6.48}$$

where

$$\lambda_1 = \frac{\bar{b}z^2 + (\bar{a}\bar{\alpha} + \bar{b}\alpha)(2z + z_0)}{2(\bar{a} + \bar{b}w)}, \; z_0 = \alpha - \bar{\alpha}w, \tag{6.49}$$

or

$$\lambda_1 = \frac{\bar{b}(z + z_0)^2}{2(\bar{a} + \bar{b}w)} + \bar{\alpha}(z + \frac{z_0}{2}). \tag{6.50}$$

Note the expression (6.48)-(6.50) *is identical with the expression given in* Theorem 1.4 *in* [28] *of the Jacobi forms, under the the identification of* $c, d, \tau, z, \mu, \lambda$ *in* [28] *with, respectively,* $\bar{b}, \bar{a}, w, z, \alpha, -\bar{\alpha}$ *in our notation. Note also that the composition law* (6.47) *of the Jacobi group* G^J *and the action of the Jacobi group on the base manifold* (4.2b) *is similar with that in the paper* [18]. See also §9 and the Corollary 3.4.4 in [19].

7. The symmetric Fock space

We recall the construction (2.6) of the map

$$\Phi : \mathcal{H}^* \rightarrow \mathcal{F}_{\mathcal{H}}; \Phi(\psi) = f_\psi,\ \ f_\psi(z) := (e_{\bar{z}}, \psi)_{\mathcal{H}},$$

and the isometric embedding (2.8).

Knowing the symmetric Fock spaces associated to the groups HW and $\mathrm{SU}(1,1)$, we shall construct in this section the symmetric Fock space associated to the Jacobi group.

We begin recalling the construction for

7.1. **The Heisenberg-Weyl group.** In the orthonormal base (5.9), Perelomov's CS-vectors associated to the HW group, defined on $M := H_1/\mathbb{R} = \mathbb{C}$, are

$$e_z := e^{za^+}e_0 = \sum \frac{z^n}{(n!)^{1/2}}|n>, \tag{7.1}$$

and their corresponding holomorphic functions are (see e.g. [3])

$$f_{|n>}(z) := (e_{\bar{z}}, |n>) = \frac{z^n}{(n!)^{1/2}}. \tag{7.2}$$

The reproducing kernel $K : \mathbb{C} \times \bar{\mathbb{C}} \rightarrow \mathbb{C}$ is

$$K(z, \bar{z}') := (e_{\bar{z}}, e_{\bar{z}'}) = \sum f_{|n>}(z)\bar{f}_{|n>}(z') = e^{z\bar{z}'}, \tag{7.3}$$

where the vector e_z is given by (7.1), while the function $f_{|n>}(z)$ is given by (7.2). In order to obtain the equality (7.3) with e_z given by (7.1), equation (5.9) is used.

The scalar product (2.1) on the Segal-Bargmann-Fock space is (cf. [3])

$$(\phi, \psi)_{\mathcal{H}^*} = (f_\phi, f_\psi)_{\mathcal{F}_{\mathcal{H}}} = \frac{1}{\pi}\int \bar{f}_\phi(z) f_\psi(z) e^{-|z|^2} d\Re z d\Im z.$$

Now we recall the similar construction for

7.2. **The group $\mathbf{SU}(1,1)$.** In the orthonormal base (5.7), Perelomov's CS-vectors for $\mathrm{SU}(1,1)$, based on the unit disk $\mathcal{D}_1 = \mathrm{SU}(1,1)/\mathrm{U}(1)$, are

$$e_z := e^{z\boldsymbol{K}_+}e_0 = \sum \frac{z^n \boldsymbol{K}_+^n}{n!}e_0 = \sum \frac{z^n e_{k,k+n}}{n! a_{kn}}, \tag{7.4}$$

and the corresponding holomorphic functions are (see e.g. equation 9.14 in [2])

$$f_{e_{k,k+n}}(z) := (e_{\bar{z}}, e_{k,k+n}) = \sqrt{\frac{\Gamma(n+2k)}{n!\Gamma(2k)}}z^n. \tag{7.5}$$

The reproducing kernel $K : \mathcal{D}_1 \times \bar{\mathcal{D}}_1 \rightarrow \mathbb{C}$ is

$$K(z, \bar{z}') := (e_{\bar{z}}, e_{\bar{z}'}) = \sum f_{e_{k,k+m}}(z)\bar{f}_{e_{k,k+m}}(z') = (1 - z\bar{z}')^{-2k}, \tag{7.6}$$

where the vector e_z is given by (7.4), while the function $f_{e_{k,k+m}}(z)$ is given by (7.5). In order to obtain the equality (7.6) for e_z given by (7.4), the orthonormality given by (5.7) is used, while for the second equality involving the functions (7.5), use is made of equation (5.8).

The scalar product (2.1) on $\mathcal{D}_1 = \mathrm{SU}(1,1)/\mathrm{U}(1)$ is (see e.g. equation 9.9 in [2])

$$(\phi,\psi)_{\mathcal{H}^*} = (f_\phi, f_\psi)_{\mathcal{F}_\mathcal{H}} = \frac{2k-1}{\pi}\int_{|z|<1} \bar{f}_\phi(z) f_\psi(z)(1-|z|^2)^{2k-2} d\Re z d\Im z, 2k = 2,3,....$$

7.3. **The Jacobi group.** In formula (4.3) defining Perelomov's CS vectors for the Jacobi group (6.41), we take into account (5.4a), (5.6a) and we have

$$e_{z,w} = \exp(za^+ + \frac{1}{2}(a+)^2 w)\exp(w\boldsymbol{K}'_+)e_0.$$

With (5.13), (5.11), we have

$$e_{z,w} = \sum_n \frac{i^{-n}}{n!}(\frac{w}{2})^{\frac{n}{2}} H_n(\frac{iz}{\sqrt{2w}})(a^+)^n \sum_m \frac{w^m}{m!}(\boldsymbol{K}'_+)^m e_0.$$

Now we take into account (5.7) and we get

$$e_{z,w} = \sum_n \frac{i^{-n}}{(n!)^{1/2}}|n>(\frac{w}{2})^n H_n(\frac{iz}{\sqrt{2w}})\sum_m \frac{w^m}{m! a_{k'm}} e_{k',k'+m}.$$

The base of functions associated to the CS-vectors attached to the Jacobi group (6.41), based on the manifold M (4.2b)

$$f_{|n>;e_{k',k'+m}}(z,w) := (e_{\bar{z},\bar{w}}, |n> e_{k',k'+m}),\ z\in\mathbb{C},\ |w|<1, \tag{7.7}$$

consists of the functions

$$f_{|n>;e_{k',k'+m}}(z,w) = (n!)^{-1/2}(\frac{i}{\sqrt{2}})^n \sqrt{\frac{\Gamma(m+2k')}{m!\Gamma(2k')}} w^{m+\frac{n}{2}} H_n(\frac{-iz}{\sqrt{2w}}). \tag{7.8}$$

Using the equation (5.12), we can write down

$$(\frac{i}{\sqrt{2}})^n w^{\frac{n}{2}} H_n(\frac{-iz}{\sqrt{2w}}) := P_n(z,w), \tag{7.9}$$

where the polynomials $P_n(z,w)$ have the expression

$$P_n(z,w) = n!\sum_{k=0}^{[\frac{n}{2}]}(\frac{w}{2})^k \frac{z^{n-2k}}{k!(n-2k)!}. \tag{7.10}$$

With the notation (7.9), equation (7.8) can be written down as

$$f_{|n>;e_{k',k'+m}}(z,w) = f_{e_{k',k'+m}}(w)\frac{P_n(z,w)}{\sqrt{n!}}, \tag{7.11}$$

where the functions $f_{e_{k',k'+m}}$ are defined in (7.5).

Above we have $2k = 2k' + 1/2$ and $m = 0,1,\ldots$.

In order to illustrate (7.10), we present the first 6 polynomials $P_n(z,w)$:

$$\begin{array}{ll} P_0 = 1; & P_1 = z; \\ P_2 = z^2 + w; & P_3 = z^3 + 3zw; \\ P_4 = z^4 + 6z^2 w + 3w^2; & P_5 = z^5 + 10z^3 w + 15zw. \end{array} \tag{7.12}$$

The reproducing kernel (5.3) $K : M \times \bar{M} \rightarrow \mathbb{C}$ has the property:

$$\begin{aligned}
(7.13a)\; K(z,w;\bar{z},\bar{w}') &:= (e_{\bar{z},\bar{w}}, e_{\bar{z}',\bar{w}'}) = \sum_{n,m} f_{|n>,e_{k',k'+m}}(z,w)\bar{f}_{|n>,e_{k',k'+m}}(z',w') \\
(7.13b)\qquad &= (1-w\bar{w}')^{-2k}\exp\frac{2\bar{z}'z+z^2\bar{w}'+\bar{z}'^2w}{2(1-w\bar{w}')}.
\end{aligned}$$

The fact that the coherent state vectors (4.3) have the scalar product given by (7.13b) was proved in Lemma 2, equation (5.3). In order to check up the equality (7.13) for the functions (7.8), we use the form (7.11), sum up the part corresponding to the functions (7.5) using the summation formula (7.6) for k', while for the part corresponding to the mixed part $P_n(z,w)$ in (7.11), we go back with (7.9) to the representation (7.8), and we apply again the summation formula (5.14) of the Hermite polynomials.

In accord with the general scheme of §2.1, the scalar product (2.1) of functions from the space $\mathfrak{F}_K$ corresponding to the kernel defined by (5.3) on the manifold (4.2b) is:

$$(\phi,\psi) = \Lambda\int_{z\in\mathbb{C};|w|<1}\bar{f}_\phi(z,w)f_\psi(z,w)(1-w\bar{w})^{2k}\exp-\frac{|z|^2}{1-w\bar{w}}\exp-\frac{z^2\bar{w}+\bar{z}^2w}{2(1-w\bar{w})}d\nu, \tag{7.14}$$

where the value of the G^J-invariant measure $d\nu$

$$d\nu = \frac{d\Re w d\Im w}{(1-w\bar{w})^3}d\Re z d\Im z \tag{7.15}$$

will be deduced latter in (7.22), in accord with the receipt (2.2).

In order to find the value of the constant Λ in (7.14), we take the functions $\phi,\psi = 1$, we change the variable $z \rightarrow (1-w\bar{w})^{1/2}z$ and we get

$$1 = \Lambda\int_{|w|<1}(1-w\bar{w})^{2k-2}d\Re w d\Im w\int_{z\in\mathbb{C}}\exp(-|z|^2)\exp(-\frac{z^2\bar{w}+\bar{z}^2w}{2})d\Re z d\Im z.$$

We apply equations (A1), (A2) in [4]:

$$I(B,C) = \int\exp(\frac{1}{2}(z.Bz+\bar{z}.C\bar{z}))\pi^{-n}e^{-|z|^2}\prod_{k=1}^n d\Re z_k d\Im z_k = [\det(1-CB)]^{-\frac{1}{2}},$$

where B, C are complex symmetric matrices such that $|B|<1, |C|<1$. Here $n=1$, $B=-\bar{w}$, $C=-w$. So, we get

$$1 = \pi\Lambda\int_{|w|<1}(1-\bar{w}w)^{2k-5/2}d\Re w d\Im w.$$

But

$$\int_{|w|<1}\frac{d\Re w d\Im w}{(1-|w|^2)^\lambda} = \frac{\pi}{1-\lambda},\text{ where } \lambda<1,$$

and we find out the value of the constant Λ in (7.14):

$$\Lambda = \frac{4k-3}{2\pi^2}. \tag{7.16}$$

7.4. **The geometry of the manifold $\mathbb{C}\times\mathcal{D}_1$.** Now we follow the general prescription of §2.2. We calculate the Kähler potential as the logarithm of the reproducing kernel (5.3), $f := \log K$, i.e.

$$f = \frac{2z\bar{z} + z^2\bar{w} + \bar{z}^2 w}{2(1-w\bar{w})} - 2k\log(1-w\bar{w}). \tag{7.17}$$

The Kähler two-form ω is given by the formula:

$$-i\omega = f_{z\bar{z}}dz\wedge d\bar{z} + f_{z\bar{w}}dz\wedge d\bar{w} - f_{\bar{z}w}d\bar{z}\wedge dw + f_{w\bar{w}}dw\wedge d\bar{w}. \tag{7.18}$$

The volume form is:

$$-\omega\wedge\omega = 2\begin{vmatrix} f_{z\bar{z}} & f_{z\bar{w}} \\ f_{\bar{z}w} & f_{\bar{w}w}\end{vmatrix} dz\wedge d\bar{z}\wedge dw\wedge d\bar{w}. \tag{7.19}$$

We start calculating the partial derivatives of the function f. We have

$$f_{\bar{z}} = \frac{z+\bar{z}w}{1-w\bar{w}},$$

$$f_{\bar{w}} = \frac{z^2 + w(2z\bar{z}+\bar{z}^2 w)}{2(1-w\bar{w})^2} + \frac{2kw}{1-w\bar{w}},$$

$$f_{w\bar{w}} = \frac{A''}{2(1-w\bar{w})^2} + \frac{2k}{(1-w\bar{w})^2},$$

where A'' is

$$A'' = (2\bar{z}^2 w + 2z\bar{z})(1-w\bar{w})^2 + 2\bar{w}(1-w\bar{w})(\bar{z}^2w^2 + 2z\bar{z}w + z^2) = (1-w\bar{w})A'.$$

Here A' is

$$A' = 2\bar{z}^2 w + 2z\bar{z} - 2\bar{z}^2 w^2\bar{w} - 2z\bar{z}\bar{w}w + 2\bar{z}^2\bar{w}w^2 + 4z\bar{z}w\bar{w} + 2z^2\bar{w}.$$

and we have finally

$$A' = 2(\bar{z}+\bar{w}z)(z+w\bar{z}).$$

So, *we find for the manifold* (4.2) *the fundamental two-form* ω (7.18), *where*

$$f_{z\bar{z}} = \frac{1}{1-w\bar{w}}, \tag{7.20a}$$

$$f_{z\bar{w}} = \frac{z+w\bar{z}}{(1-w\bar{w})^2}, \tag{7.20b}$$

$$f_{w\bar{w}} = \frac{(\bar{z}+\bar{w}z)(z+w\bar{z})}{(1-w\bar{w})^3} + \frac{2k}{(1-w\bar{w})^2}. \tag{7.20c}$$

We can write down the two-form ω (7.18)-(7.20) as

$$-i\omega = \frac{2k}{(1-w\bar{w})^2}dw\wedge d\bar{w} + \frac{A\wedge\bar{A}}{1-w\bar{w}},\ A = dz + \bar{\alpha}_0 dw,\ \alpha_0 = \frac{z+\bar{z}w}{1-w\bar{w}}. \tag{7.21}$$

For the volume form (7.19), we find:

$$\omega\wedge\omega = 4k(1-w\bar{w})^{-3}4\Re z\Im z\Re w\Im w. \tag{7.22}$$

It can be checked up that indeed, *the measure $d\nu$ and the fundamental two-form ω are group-invariant at the action* (6.43) *of the Jacobi group* (6.41).

Now we summarize the contents of this section as follows:

Proposition 2. *Let us consider the Jacobi group G^J_1 (6.41) with the composition rule (6.47) acting on the coherent state manifold (4.2) via equation (6.43). The manifold $\mathcal{D}^J_1$ has the Kähler potential (7.17) and the G^J_1-invariant Kähler two-form ω given by (7.21). The holomorphic polynomials (7.7) associated to the coherent state vectors (4.3) are given by (7.11), where the functions f are given by (7.5), while the polynomials P are given by (7.10). The Hilbert space of holomorphic functions $\mathcal{F}_K$ associated to the holomorphic kernel $K : \mathcal{D}^J_1 \times \bar{\mathcal{D}}^J_1 \rightarrow \mathbb{C}$ given by (5.3) is endowed with the scalar product (7.14), where the normalization constant Λ is given by (7.16) and the G^J_1-invariant measure $d\nu$ is given by (7.15).*

Recalling Proposition IV.1.9. p. 104 and Proposition XII.2.1 p. 515 in [52], Proposition 1 can be formulated as follows:

Proposition 3. *Let $h := (g, \alpha) \in G^J_1$, where G^J_1 is the Jacobi group (6.41), and we consider the representation $\pi(h) := S(g)D(\alpha)$, $g \in \mathrm{SU}(1,1)$, $\alpha \in \mathbb{C}$, and let the notation $x := (z, w) \in \mathcal{D}^J_1 := \mathbb{C} \times \mathcal{D}_1$. Then the continuous unitary representation $(\pi_K, \mathcal{H}_K)$ attached to the positive definite holomorphic kernel K defined by (5.3) is*

$$(\pi_K(h).f)(x) = J(h^{-1}, x)^{-1} f(h^{-1}.x), \tag{7.23}$$

where the cocycle $J(h^{-1}, x)^{-1} := \lambda(h^{-1}, x)$ with λ defined by equations (6.42)-(6.46) and the function f belongs to the Hilbert space of holomorphic functions $\mathcal{H}_K \equiv \mathcal{F}_K$ endowed with the scalar product (7.14), where Λ is given by (7.16).

Remark 10. *The equations of the geodesics on the manifold $\mathcal{D}^J_1$ endowed with the two-form (7.21) in the variables $w \in \mathcal{D}_1, z \in \mathbb{C}$ are*

$$2k\frac{d^2 z}{dt^2} - \bar{\alpha}_0(\frac{dz}{dt})^2 + 2(2k\frac{\bar{w}}{P} - \bar{\alpha}^2_0)\frac{dz}{dt}\frac{dw}{dt} - \bar{\alpha}^3_0(\frac{dw}{dt})^2 = 0; \tag{7.24a}$$

$$2k\frac{d^2 w}{dt^2} + (\frac{dz}{dt})^2 + 2\bar{\alpha}_0\frac{dz}{dt}\frac{dw}{dt} + (4k\frac{\bar{w}}{P} + \bar{\alpha}^2_0)(\frac{dw}{dt})^2 = 0, \tag{7.24b}$$

where α_0 is given by (6.45) and $P = 1 - w\bar{w}$.

8. Physical applications: classical and quantum equations of motion

We consider applications of the formulas (4.4) proved in this paper for the Jacobi group G^J_1 to equations of motion on the CS-manifold $\mathcal{D}^J_1$. This extend our previous results for hermitian groups [7, 8] or semisimple groups which generate CS-orbits [10] to an example of non-reductive CS-group.

Passing on from the dynamical system problem in the Hilbert space $\mathcal{H}$ to the corresponding one on M is called sometimes *dequantization*, and the system on M is a classical one [7],[8]. Following Berezin [16], the motion on the classical phase space can be described by the local equations of motion

$$\dot{z}_\gamma = i\{\boldsymbol{\mathcal{H}}, z_\gamma\}, \tag{8.1}$$

where $\boldsymbol{\mathcal{H}}$ is the energy function attached to the Hamiltonian $\boldsymbol{H}$. In (8.1) $\{\cdot, \cdot\}$ denotes the Poisson bracket:

$$\{f, g\} = \sum_{\alpha,\beta\in\Delta_+} \mathcal{G}^{-1}_{\alpha,\beta}\left\{\frac{\partial f}{\partial z_\alpha}\frac{\partial g}{\partial \bar{z}_\beta} - \frac{\partial f}{\partial \bar{z}_\alpha}\frac{\partial g}{\partial z_\beta}\right\}, f, g \in C^\infty(M),$$

where $\mathcal{G}$ are defined in (2.4). The equations of motion (8.1) can be written down as

$$i\begin{pmatrix} 0 & \mathcal{G} \\ -\bar{\mathcal{G}} & 0 \end{pmatrix}\begin{pmatrix} \dot{z} \\ \dot{\bar{z}} \end{pmatrix} = -\begin{pmatrix} \frac{\partial}{\partial z} \\ \frac{\partial}{\partial \bar{z}} \end{pmatrix}\boldsymbol{\mathcal{H}}. \tag{8.2}$$

We consider an algebraic Hamiltonian linear in the generators of the group of symmetry

$$\boldsymbol{H} = \sum_{\lambda\in\Delta}\epsilon_\lambda \boldsymbol{X}_\lambda. \tag{8.3}$$

We recall (cf. [7, 8]) that *if the differential action of the generators of the group G is given by formulas* (2.11)*, then classical motion and the quantum evolutions generated by the Hamiltonian* (8.3) *are given by the same equations of motion* (8.1) *on $M = G/H$:*

$$i\dot{z}_\alpha = \sum_\lambda \epsilon_\lambda Q_{\lambda,\alpha}. \tag{8.4}$$

The two-form form ω on M permits to determine the Berry phase [8].

Let us consider a linear Hamiltonian in the generators of the Jacobi group (6.41):

$$\boldsymbol{H} = \epsilon_a a + \bar{\epsilon}_a a^+ + \epsilon_0 \boldsymbol{K}_0 + \epsilon_+ \boldsymbol{K}_+ + \epsilon_- \boldsymbol{K}_-. \tag{8.5}$$

We have

Remark 11. *The equations of motion on the manifold* (4.2b) *generated by the linear Hamiltonian* (8.5) *are given by the matrix Riccati equation:*

$$i\dot{z} = \epsilon_a + \frac{\epsilon_0}{2}z + \epsilon_+ zw, \tag{8.6a}$$

$$i\dot{w} = \epsilon_- + (\bar{\epsilon}_a + \epsilon_0)w + \epsilon_+ w^2. \tag{8.6b}$$

Note that the second equation (8.6b) is a Riccati equation on $\mathcal{D}_1$. The procedure of linearization of matrix Riccati equation on manifolds is discussed in [8].

An interesting development of the present construction should be to consider nonlinear CSs [62] attached to a deformed Jacobi group. However, difficulties in the physical interpretation of the creation and annihilation of q-deformed oscillator, related to the quantum groups $\mathrm{SU}_q(2)$ [22], appear as these are symmetric but not self-adjoint operators [45].

9. Comparison with Kähler-Berndt's approach

Rolf Berndt -alone or in collaboration - has studied the real Jacobi group $G^J(\mathbb{R})$ in several references, from which I mention [17, 18, 19, 20]. The Jacobi group appears (see explanation in [40]) in the context of the so called *Poincaré group* or *The New Poincaré group* investigated by Erich Kähler as the 10-dimensional group G^K which invariates a hyperbolic metric [37, 38, 39]. Kähler and Berndt have investigated the Jacobi group $G^J_0(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ acting on the manifold $\mathcal{X}^J_1 := \mathcal{H}_1 \times \mathbb{C}$, where $\mathcal{H}_1$ is the upper half plane $\mathcal{H}_1 := \{v \in \mathbb{C} | \Im(v) > 0\}$.

For self-contentedness, in Remarks 12 and 13 below, we briefly proof two results from [19], which we need in order two express our two form ω in the coordinates used by Kähler and Berndt.

Remark 12. *The action of* $G^J_0(\mathbb{R})$ *on* $\mathcal{X}^J_1$ *is given by* $(g,(v,z)) \rightarrow (v_1,z_1)$, $g=(M,l)$, *where*

$$v_1 = \frac{av+b}{cv+d}, z_1 = \frac{z+l_1v+l_2}{cv+d};\ M = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \mathrm{SL}_2(\mathbb{R}), (\mathrm{l}_1,\mathrm{l}_2) \in \mathbb{R}^2. \tag{9.1}$$

Proof. Let us use the notation of [19]. We denote $G^J(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R}) \ltimes H(\mathbb{R})$, where $H(\mathbb{R})$ is the real HW group with the composition law:

$$(\lambda,\mu,\kappa)(\lambda',\mu',\kappa') = (\lambda+\lambda',\mu+\mu',\kappa+\kappa'+\left|\begin{array}{c} X \\ X' \end{array}\right|), \left|\begin{array}{c} X \\ X' \end{array}\right| = \det\left(\begin{array}{c} X \\ X' \end{array}\right). \tag{9.2}$$

If $g=(M,X,\kappa) \in G^J(\mathbb{R})$, where $M \in \mathrm{SL}_2(\mathbb{R})$, $X=(\lambda,\mu)$, $(X,\kappa) \in \mathbb{R}^3$, then the composition law in the real Jacobi group is

$$gg' = (MM', XM'+X', \kappa+\kappa'+\left|\begin{array}{c} XM' \\ X' \end{array}\right|). \tag{9.3}$$

The action of $G^J(\mathbb{R})$ over the $H(\mathbb{R})$ is

$$M(X,\kappa)M^{-1} = (XM^{-1},\kappa). \tag{9.4}$$

Let us consider the Iwasawa decomposition for a matrix $M \in \mathrm{SL}_2(\mathbb{R})$:

$$M = \left(\begin{array}{cc} 1 & x \\ 0 & 1 \end{array}\right)\left(\begin{array}{cc} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{array}\right)\left(\begin{array}{cc} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{array}\right),\ y>0. \tag{9.5}$$

If

$$M = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right), \tag{9.6}$$

then we find for x,y,θ in (9.5)

$$x = \frac{ac+bd}{d^2+c^2};\ y = \frac{1}{d^2+c^2};\ \sin\theta = -\frac{c}{\sqrt{c^2+d^2}};\ \cos\theta = \frac{d}{\sqrt{c^2+d^2}}. \tag{9.7}$$

For $g=(M,X,\kappa) \in G^J(\mathbb{R})$, the EZ-coordinates (Eichler-Zagier, cf. the definition at p. 12 and p. 51 in [19]) are $(x,y,\theta,\lambda,\mu,\kappa)$. Let $\tau = x+iy \in \mathcal{H}_1$, $z=\xi+i\eta = p\tau+q$, where

$$(p,q) = XM^{-1} = (\lambda d-\mu c, -\lambda b+\mu a) \tag{9.8}$$

Using the multiplication law (9.3), the Iwasawa decomposition (9.5) and the equations (9.7), (9.8), we find the action of $G^J(\mathbb{R})$ on the base $\mathcal{X}^J_1$

$$g(\tau,z) = (\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\mu}{c\tau+d}), \tag{9.9}$$

and Remark 12 is proved. ■

Let us now recall that

$$C^{-1}\mathrm{SL}_2(\mathbb{R})C = \mathrm{SU}(1,1),\ \text{where } C = \left(\begin{array}{cc} i & i \\ -1 & 1 \end{array}\right). \tag{9.10}$$

If $M \in \mathrm{SL}_2(\mathbb{R})$ is the matrix (9.6), then, under the transformation (9.10)

$$M^* = C^{-1}MC = \left(\begin{array}{cc} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{array} \right), \; \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1, \tag{9.11}$$

where

$$2\alpha = a + d + i(b - c); \; 2\beta = a - d - i(b + c). \tag{9.12}$$

Now we pass to the complex group $G^J_{\mathbb{C}} = C^{-1}G^J(\mathbb{R})C$. We recall that the Jacobi group $G^J_{\mathbb{C}}$ is a *group of Harish-Chandra type*, (cf. e.g. p. 514 in [52]; see the definition in Ch. III §5 in [59] and Ch. XII.1 in [52]). Moreover, it is well known that *the Jacobi algebra* (3.4) *is a CS-Lie algebra* (cf. e.g. Theorem 5.2 in [48]). The correspondence between our notation and that of Berndt-Schmidt at p. 12 in [19] is as follows: $a^+, a, K_+, K_-, 1, K_0$ corresponds, respectively to: $Y_+, Y_-, X_+, -X_-, -Z_0, \frac{1}{2}Z$. See also our Remark 16 below.

We see that under the transformation (9.10), $g = (M, X, \kappa) \in \mathrm{SL}_2(\mathbb{R}) \ltimes H(\mathbb{R})$ is twisted to $g^* = (M^*, X^*, \kappa)$, where M^* is given by (9.11), while, due to action (9.4), $X^* = XC = (i\lambda - \mu, i\lambda + \mu)$.

Also the map (9.10) induces a transformation of the bounded domain $\mathcal{D}_1$ into the upper half plane $\mathcal{H}_1$ and

$$\tau \in \mathcal{H}_1 \mapsto \tau^* = C^{-1}(\tau) = \frac{\tau - i}{\tau + i} \in \mathcal{D}_1. \tag{9.13}$$

The action $C^{-1}G^J_0(\mathbb{R})C$ descends on the basis as the biholomorphic map: $\check{C}^{-1} : \mathcal{X}^J_1 := \mathcal{H}_1 \times \mathbb{C} \rightarrow \mathcal{D}^J_1 := \mathcal{D}_1 \times \mathbb{C}$: $(\tau, z) \mapsto (\tau^*, z^*)$. Here $\tau*$ is given by (9.13), while $z^* = p^*\tau^* + q^*$. So, $(p, q) = (\lambda, \mu)M^{-1}$, and $(p^*, q^*) = (\lambda^*, \mu^*)M^{*-1}$. But $M^* = C^{-1}MC$, and $(p^*, q^*) = (p, q)C = (-q + ip, q + ip)$, and we get $z^* = \frac{2iz}{\tau+i}$. Note that at p. 53 in [19] the factor $2i$ in this formula is missing.

In a different notation, we have shown that

Remark 13. *The action $C^{-1}G^J_0(\mathbb{R})C$, descends on the basis as the biholomorphic map: $\check{C}^{-1} : \mathcal{X}^J_1 := \mathcal{H}_1 \times \mathbb{C} \rightarrow \mathcal{D}^J_1 := \mathcal{D}_1 \times \mathbb{C}$:*

$$w = \frac{v - i}{v + i}; \; z = \frac{2iu}{v + i}, w \in \mathcal{D}_1, \; v \in \mathcal{H}_1, z \in \mathbb{C}. \tag{9.14}$$

The $G^J_0(\mathbb{R})$-invariant closed two-form considered by Kähler-Berndt is:

$$\omega' = \alpha\frac{dv \wedge d\bar{v}}{(v - \bar{v})^2} + \beta\frac{1}{v - \bar{v}}B \wedge \bar{B}, \; B = du - \frac{u - \bar{u}}{v - \bar{v}}dv, v, u \in \mathbb{C}, \; \Im(v) > 0, \tag{9.15}$$

cf.§36 in [39]; see also §3.2 in [17], where the first term is misprinted as $\alpha\frac{dv\wedge d\bar{v}}{v-\bar{v}}$.

Under the mapping (9.14), the two-form ω (7.21) reads

$$-i\omega = -\frac{2k}{(\bar{v} - v)^2}dv \wedge d\bar{v} + \frac{2}{i(\bar{v} - v)}B \wedge \bar{B}, \tag{9.16}$$

i.e. (9.15). In fact, we have proved that

Remark 14. *When expressed in the coordinates $(v, u) \in \mathcal{X}^J_1$ which are related to the coordinates $(w, z) \in \mathcal{D}^J_1$ by the map* (9.14) *given by* Remark 13*, our Kähler two-form* (7.21) *is identical with the one* (9.16) *considered by Kähler-Berndt.*

If we use the EZ coordinates adapted to our notation

$$v = x + iy;\ u = pv + q,\ x, p, q, y \in \mathbb{R}, y > 0, \tag{9.17}$$

the $G_0^J(\mathbb{R})$-invariant Kähler metric on $\mathcal{X}^J$ corresponding to the Kähler-Berndt's Kähler two-form ω (9.16) reads

$$ds^2 = \frac{k}{2y^2}(dx^2 + dy^2) + \frac{1}{y}[(x^2 + y^2)dp^2 + dq^2 + 2xdpdq], \tag{9.18}$$

i.e. the equation at p. 62 in [19] or the equation given at p. 30 in [17].

The Kähler two-form (9.15) of Kähler-Berndt corresponds (cf. equation (9) in Ch. 36 of [37]) to the Kähler potential

$$f' = -\frac{\lambda}{2}\log\frac{v - \bar{v}}{2i} - i\pi\mu\frac{(u - \bar{u})^2}{v - \bar{v}}, u \in \mathbb{C},\ v \in \mathcal{H}_1. \tag{9.19}$$

The Kähler potential (9.19), corresponds to some *(Kähler) Perelomov's CS-vectors based on the CS-manifold* $\mathcal{X}_1^J$ on which the group $G_0^J(\mathbb{R})$ acts via the the action (9.1), which, instead of the scalar product K (5.2), should have *a scalar product* K' *in the EZ coordinates* (9.17) $x, y, p, q \in \mathbb{R}$

$$K' = y^{-\frac{\lambda}{2}}\exp(2\pi\mu p^2 y). \tag{9.20}$$

It would be interesting to extend eq. (9.16) to the case $H_n \rtimes \mathrm{Sp}(n, \mathbb{R})$, see reference [14]. This would be useful for a better understanding of the Gaussons [60, 1].

10. Some more comments

Remark 15. We have called the algebra (3.4), the *Jacobi algebra* and the group (6.41), the *Jacobi group*, in agreement with the name used in [19] or at p. 178 in [52], where the algebra $\mathfrak{g}_1^J := \mathfrak{h}_1 \rtimes \mathfrak{sl}(2, \mathbb{R})$ is called "Jacobi algebra". The denomination adopted in the present paper is of course in accord with the one used in [52] because of the isomorphism of the Lie algebras $\mathfrak{su}(1,1) \sim \mathfrak{sl}(2,\mathbb{R}) \sim \mathfrak{sp}(1,\mathbb{R})(\sim \mathfrak{so}(2,1))$. Also the name "Jacobi algebra" is used in [52] p. 248 to call the semi-direct sum of the $(2n+1)$-dimensional Heisenberg algebra and the symplectic algebra, $\mathfrak{hsp} := \mathfrak{h}_n \rtimes \mathfrak{sp}(n, \mathbb{R})$. The group corresponding to this algebra is called sometimes in the Mathematical Physics literature (see e. g. §10.1 in [1], which is based on [60]) the "metaplectic group", but in reference [52] the term "metaplectic group" is reserved to the 2-fold covering group of the symplectic group, cf. p. 402 in [52] (see also [4] and [35]). Other names of the metaplectic representation are the oscillator representation, the harmonic representation or the Segal-Shale-Weil representation, see references in Chapter 4 of [29] and [19].

Remark 16. We now discuss the Jacobi algebra (3.4) from the view point of the book [52]. We know (cf. Lemma XII.1.20 p. 509) that *quasihermitian* Lie algebras, i.e. Lie algebras for which a *maximal compactly embedded subalgebra* $\mathfrak{k}$ (cf. Definition VII.1.1 p. 222 in [52]) satisfies the relation $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{z}(\mathfrak{k})) = \mathfrak{k}$ (cf. Definition VII.2.15 p. 241 in [52]), admit the 5-grading of the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}_s^+ \oplus \mathfrak{p}_r^+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_r^- \oplus \mathfrak{p}_s^-$, where $\mathfrak{k}$ is the maximal compactly embedded subalgebra of $\mathfrak{g}$, $\mathfrak{p}_s$ ($\mathfrak{p}_r$) represent the semisimple roots, (respectively, the solvable roots) (cf. Definition VII.2.4 p. 234 in [52]), while "+" ("−") refers to the positive (respectively, negative) roots with respect to a Δ^+ adopted positive

system (cf. Definition VII.2.6 p. 236 in [52]). But *the Jacobi algebra is quasihermitian*, cf. Example VIII.2.3 p. 294 in [52], with $\mathfrak{t} = \{0\} \oplus \mathbb{R} \oplus \mathfrak{u}(1)$ a compactly embedded Cartan algebra and also a maximal compactly embedded subalgebra of $\mathfrak{g}^J$ (cf. Example VII.2.30 p. 249 and Example XII.1.22 p. 513 in [52]). So, the generators K_+, K_-, a^+, a of the Jacobi algebra (3.4) belong to $\mathfrak{p}^+_s$, $\mathfrak{p}^-_s$, $\mathfrak{p}^+_r$, respectively $\mathfrak{p}^-_r$, 1 belongs to the $\mathbb{R}$-part of $\mathfrak{k}$, while K_0 belongs to the $\mathfrak{u}(1)$-part of $\mathfrak{k}$. Note that due to relation (3.5d), the subalgebra $\mathfrak{p}^+ = \mathfrak{p}^+_r \oplus \mathfrak{p}^+_s$ which appears in the definition (4.3) of Perelomov's coherent state vectors is an abelian one, as it should be (cf. Lemma XII.1.20 p. 509 in [52]).

Remark 17. We emphasize that *the representation given in* Lemma 1 *is different from the extended metaplectic representation* (cf. p. 196 in [29], see also [59]). As was already mentioned, the Jacobi algebra admits a realization as subalgebra of the Weyl algebra A_1 of polynomials in p, q of degree ≤ 2. In the present paper we have presented a realization of the Jacobi algebra as subalgebra of the Weyl algebra A_2 defined by holomorphic first order differentials operators with holomorphic polynomials of degree ≤ 2. This algebra is realized in the variables $x = (z, w) \in \mathcal{D}^J_1$. We recall that the only finite-dimensional non-solvable Lie algebras that can be realized as Lie subalgebras of the complex Weyl algebra A_1 are: $\mathfrak{sp}(1,\mathbb{R})$, $\mathfrak{sp}(1,\mathbb{R}) \times \mathbb{C}$ and the Jacobi algebra [61], [36], [57].

Remark 18. Note that the expression (5.3) of the reproducing kernel is the particular case $n = 1$, $2k = \frac{1}{2}$ of the reproducing kernel on the space $\mathcal{D}^J_n := \mathbb{C}^n \times \mathcal{D}_n$, where $\mathcal{D}_n$ is the Siegal ball, at p. 532 in the book of K.-H. Neeb [52] or in the article [32] and (5.28) in the book [59]. See also [14].

11. Appendix

For self-completeness, we also give

Proof of (5.9). We shall calculate

$$\lambda_{n;m} := (e_0, a^n (a^+)^m e_0).$$

We shall use the formula

$$[A, B^m] = \sum_{s=0}^{m-1} B^s [A, B] B^{m-s-1}, \tag{11.1}$$

for $A = a^n$, $B = a^+$. We have

$$[A, B^m] = -\sum_{s=0}^{m-1} B^s [B, A] B^{m-s-1}.$$

But

$$[B, A] = [a^+, a^n] = \sum_{p=0}^{n-1} a^p [a^+, a] a^{n-p-1} = -\sum_{p=0}^{n-1} a^{n-1} = -n a^{n-1},$$

and

$$[A, B^m] = n\sum_{s=0}^{m-1}(a^+)^s a^{n-1}(a^+)^{m-s-1},$$

so, we have

$$\lambda_{n;m} = n\lambda_{n-1;m-1}.$$

If $n = m$, then $\lambda_{nn} = n!$. If $n < m$, then $\lambda_{n;m} = ct(e_0, [a, (a^+)^p]e_0)$, where $p > 1$. But $[a, (a^+)^p] = \sum_{s=0}^{p-1}(a^+)^s[a, a^+](a^+)^{p-s-1} = p(a^+)^{p-1}$, and $\lambda_{n;m} = ct(e_0, (a^+)^{p-1}e_0) = 0$.

Similarly, if $n > m$, then $[a^p, a^+] = pa^{p-1}$ and also $\lambda_{n;m} = ct(e_0, a^{p-1}e_0) = 0$.

So, we have $\lambda_{n;m} = n!\delta_{n;m}$. ■

Proof of (5.7). We calculate $\mu_{n;m} := (e_0, C_{n;m}e_0)$, where $C_{n;m} = \boldsymbol{K}_-^n\boldsymbol{K}_+^m$, using (11.1) with $A = \boldsymbol{K}_-^n$ and $B = \boldsymbol{K}_+$. We find

$$\mu_{n;m} = (e_0, \sum_{s=0}^{m-1}\boldsymbol{K}_+^s[\boldsymbol{K}_-^n, \boldsymbol{K}_+]\boldsymbol{K}_+^{m-s-1}e_0) = (e_0, [\boldsymbol{K}_-^n, \boldsymbol{K}_+]\boldsymbol{K}_+^{m-1}e_0).$$

But

$$\begin{aligned}
[\boldsymbol{K}_+, \boldsymbol{K}_-^n] &= \sum_{p=0}^{n-1}\boldsymbol{K}_-^p[\boldsymbol{K}_+, \boldsymbol{K}_-]\boldsymbol{K}_-^{n-p-1} = -2\sum_{p=0}^{n-1}\boldsymbol{K}_-^p\boldsymbol{K}_0\boldsymbol{K}_-^{n-p-1}\\
&= -2\sum_{p=0}^{n-1}\boldsymbol{K}_-^p[\boldsymbol{K}_0, \boldsymbol{K}_-^{n-p-1}] - 2\sum_{p=0}^{n-1}\boldsymbol{K}_-^p\boldsymbol{K}_-^{n-p-1}\boldsymbol{K}_0\\
&= -2n\boldsymbol{K}_-^{n-1}\boldsymbol{K}_0 - 2\sum_{p=0}^{n-1}\boldsymbol{K}_-^p[\boldsymbol{K}_0, \boldsymbol{K}_-^{n-p-1}].
\end{aligned}$$

We find

$$\mu_{n;m} = 2n(e_0, \boldsymbol{K}_-^{n-1}\boldsymbol{K}_0\boldsymbol{K}_+^{m-1}e_0) + R,$$

where

$$R := 2(e_0, R_0e_0);\ \ R_0 := \sum_{p=0}^{n-1}\boldsymbol{K}_-^p[\boldsymbol{K}_0, \boldsymbol{K}_-^{n-p-1}]\boldsymbol{K}_+^{m-1},$$

and we get

$$\mu_{n;m} = 2nk\mu_{n-1,m-1} + 2n(e_0, \boldsymbol{K}_-^{n-1}[\boldsymbol{K}_0, \boldsymbol{K}_+^{m-1}]e_0) + R.$$

But

$$[\boldsymbol{K}_0, \boldsymbol{K}_+^{m-1}] = \sum_{k=0}^{m-2}\boldsymbol{K}_+^s[\boldsymbol{K}_0, \boldsymbol{K}_+]\boldsymbol{K}_+^{m-2-s} = (m-1)\boldsymbol{K}_+^{m-1},$$

and

$$[\boldsymbol{K}_0, \boldsymbol{K}_-^{n-p-1}] = \sum_{q=0}^{n-p-2}\boldsymbol{K}_-^q[\boldsymbol{K}_0, \boldsymbol{K}_-]\boldsymbol{K}_-^{n-p-q-2} = -(n-p-1)\boldsymbol{K}_-^{n-p-1}.$$

We get successively

$$R_0 = -\sum_{p=0}^{n-1}(n-p-1)C_{n-1;m-1},$$
$$R = -n(n-1)\mu_{n-1;m-1},$$

and

$$\mu_{n;m} = (2nk + 2n(m-1) - n(n-1))\mu_{n-1;m-1},$$
$$\mu_{n;n} = n(2k+n-1)\mu_{n-1;n-1};\ \mu_{1;1} = 2k,$$
$$\mu_{n;n} = \frac{n!(2k+n-1)!}{(2k-1)!} = \frac{n!\Gamma(2k+n)}{\Gamma(2k)}.$$

If $n < m$, then there is a $p > 1$ such that

$$[\boldsymbol{K}_-, \boldsymbol{K}_+^p] = \sum_{q=0}^{p-1}\boldsymbol{K}_+^q[\boldsymbol{K}_-, \boldsymbol{K}_+]\boldsymbol{K}_+^{p-1-q} = 2\sum_{q=0}^{p-1}\boldsymbol{K}_+^q\boldsymbol{K}_0\boldsymbol{K}_+^{p-1-q},$$

which leads in the expression of $\mu_{n;m}$ to the term $2\boldsymbol{K}_0\boldsymbol{K}_+^{p-1}$, and, after acting to the left with $\boldsymbol{K}_0$, we get a 0 contribution.

Similarly, if $n > m$, then

$$[\boldsymbol{K}_-^p, \boldsymbol{K}_+] = -\sum_{s=0}^{p-1}\boldsymbol{K}_-^s[\boldsymbol{K}_0, \boldsymbol{K}_-]\boldsymbol{K}_-^{p-s-1},$$

and $s = p-1$ in the sum. Acting on the right with $\boldsymbol{K}_0$, the contribution is also 0 because of the action on the right with $\boldsymbol{K}_-^{p-1}$. ∎

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